STA 104 Name:____________________________

MTH 135 **Probability Final Examination**

7:00-10:00 pm Wednesday, 13 December 1995

This is a closed-book examination, so please do not refer to your notes, the text, or to any other books. There are ten questions, each counting ten points. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other. Please sign the Duke Honor Code: **I have neither given nor received aid on this examination: ____________________________________ .**

You must show some work to get credit—*unsupported answers are not acceptable.* A **normal**-distribution **table** is provided, and two blank pages to use as **work sheets**; you may also **use the back** of any of the examination pages as scratch paper or for more room to show your work. It is to your advantage to **write** your solutions as **clearly** as possible, since I can't give you credit for solutions I don't understand. Good luck.

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MTH 135 **Probability Final Examination**

1. Two coins look similar, but have different probabilities of falling "heads". One is a **Fair** coin, with $\Pr[H] = 1/2$, but the other is weighted so that $\Pr[H] = 8/10$. One of the coins is chosen at random and is tossed 10 times; **X** is the number of Heads to appear, and **F** is the event that the **fair** coin was drawn.

a. If $X = 7$ is observed, what is the probability that the coin was fair?

$\Pr[F|X = 7] =$ ____________

b. If the coin is fair, and $X \geq 9$, what is the probability that $X = 10$?

$\Pr[X = 10|X \geq 9, F] =$ ____________

2. A certain test is going to be repeated until done satisfactorily. Assume that repetitions of the test are independent and that each has probability 0.50 of being satisfactory.

a. The Acme Laboratory will perform the tests for \$55 each, regardless of the outcome. Find the expected cost C_A of having Acme run the tests until a satisfactory result is obtained:

$\mathsf{E}[C_A] =$ ____________

b. Bravo, a competing testing laboratory, also offers the test but with a different pricing schedule: the first test costs \$100 to perform, and all subsequent tests cost \$20 each, regardless of the outcome. Find the expected cost C_B of having Bravo run the tests until a satisfactory result is obtained:

$\mathsf{E}[C_B] =$ ____________

c. What is the probability that Acme would charge less than Bravo? Show your work.

$\Pr[C_A < C_B] =$ ____________

d. All other things being equal, which laboratory would you recommend? Why?

Circle one: A B Reasoning:

3. In a certain gambling game four fair coins are thrown in the air. If they match (all heads or all tails), you win \$C; if they don't, you pay \$1.

a. For the game to be "fair" in the sense that your long-run average gain (or loss) will be zero, what must be the value of \$C?

   $C =$ ____________ Why?

b. Using the value of C you found above, find the mean and variance of your net winnings W_{100} after 100 plays:

   $\mathsf{E}[W_{100}] =$ ____________ $\mathsf{V}[W_{100}] =$ ____________

c. Using the value of C you found above, find (to three correct decimal places) the probability of losing at least \$10 in 100 plays:

   $\Pr[W_{100} \leq -10] =$ ____________

4. The danger of a certain post-operative complication falls off slowly in the days following an operation, with hazard $h(t) = \frac{2}{1+t}$ for $t > 0$. Let T be the time until the complication occurs, measured in days.

a. Find the survival function:

$\Pr[T > t] =$ ____________

b. For a patient who has not had the complication in the first two weeks find the probability that it will occur within the next 24 hours:

c. For a patient who has not had the complication in the first two weeks find the conditional probability density function for the time of failure (be sure to indicate where the density function is non-zero):

$f_T(t|T > 14) =$ ____________

5. Let X and Y be two points drawn independently from the unit interval $(0,1)$ with density functions $f_X(x) = 2x$ and $f_Y(y) = 3y^2$. Find the:

a. Means:

$\mathsf{E}[X] =$ ____________ $\mathsf{E}[Y] =$ ____________

b. Probabilities:

$\Pr[X < 1/2] =$ ____________ $\Pr[Y < 1/2] =$ ____________

c. Probability:

$\Pr[X < Y] =$ ____________

6. A coin of diameter 1 inch is tossed in the air and caught in an empty soup can of bottom **diameter** 4 inches. The coin lies flat on the bottom.

a. What is the chance that the coin covers the center point of the bottom of the can?

Pr = ____________

b. If instead of the can we catch the coin in a box whose bottom is square with sides of length 4 inches, what is the probability that the coin covers the center of the box?

Pr = ____________

c. State any assumptions you made:

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7. On good days at the Flu Clinic, only 1/3 of the entering patients have flu symptoms, all independently; on bad days, 2/3 of the entering patients do, again all independently. About half the days are good and half are bad. You can't tell if the day is good or bad, except by observing the patients.

a. If the first patient shows flu symptoms, what is the probability that the second one will too?

   $\Pr[F_2|F_1] =$ ____________

b. If none of the first n patients show flu symptoms, what is the probability that it's a good day?

   $\Pr[G|{F_1}^c, ..., {F_n}^c] =$ ____________

c. On good days, successive patients are independent; on bad days, successive patients are also independent; but perhaps F_1 and F_2 are *not* independent, since observing F_1 would help us guess whether the day is good or bad. Are F_1 and F_2 positively dependent, negatively dependent, or independent?

   Circle one: $+$ $-$ Ind Work:

8. The number of patients entering a clinic each day, X, has a uniform distribution on the numbers $0, 1, 2$. About half the patients complain of flu symptoms, and about half don't. If we can assume that patients complain of flu symptoms independently, with probability $p = 1/2$ each,

a. Find the probability distribution for the number Y of flu-symptom patients per day:

   $\Pr[Y = y] =$ ____________

b. What is the probability that exactly one of the day's patients *does not* have flu symptoms?

   $\Pr[X - Y = 1] =$ ____________

c. If we observe no flu patients, about how many patients do you expect we observed altogether?

   $\mathsf{E}[X|Y = 0] =$ ____________

9. The random variables X and Y are independent, each with a normal distribution but with different means and variances. The mean and variance of X are $\mu_X = 5$ and $\sigma_X^2 = 4^2 = 16$, those of Y are $\mu_Y = 0$ and $\sigma_Y^2 = 3^2 = 9$. Set $Z = X - Y$. Find the indicated quantities ("Cov" is *covariance*, "Cor" is *correlation*); show your work.

a. What is the probability distribution of Z?

b. $\Pr[X < 0] =$ ____________

c. $\Pr[X < Y] =$ ____________

d. $\mathsf{Cov}[X, Z] =$ ____________ (Covariance)

e. $\mathsf{Cor}[X, Z] =$ ____________ (Correlation)

10. Suppose an airline accepted 12 reservations for a commuter plane with 10 seats. They know that 7 reservations went to regular customers who will show up for sure, and believe that the remaining 5 passengers will show up with a 50% chance, independently of each other.

a. Find the probability that the flight will be overbooked, ie, that more passengers will show up than the seats that are available:

b. Find the probability that there will be one or more empty seats.

c. Find the expectation of the number of passengers X who are turned away:

$\mathsf{E}[X] =$ ____________

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