STA 104 Name:______________________________

MTH 135

Probability Final Examination

9:00am-12:00n Friday, 20 December, 1996

This is a closed-book examination, so please do not refer to your notes, the text, or to any other books. You may use a *single sheet* of *your own* notes, if you wish, but you may not share materials. A normal distribution table and a blank worksheet are attached to the exam. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other. Please acknowledge the Duke Honor Code:

I have neither given nor received aid on this examination: __________________ .

You must **show** some **work** to get credit—*unsupported answers are not acceptable*, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give you credit for solutions I do not understand. You should spend about 15–20 minutes on each problem. Good luck.

1.	/10
2.	/10
3.	/10
4.	/10
5.	/10
6.	/10
7.	/10
8.	/10
9.	/10
10.	/10
	/100

STA 104
MTH 135

Probability Final Examination

1. Events A and B each have probability $1/2$, while their union has probability $\mathsf{P}[A \cup B] = 5/8$. Find:

a. The conditional probability of A, if B occurs:

$\mathsf{P}[A|B] =$ ____________________

b. The probability that A and B both occur:

$\mathsf{P}[A \cap B] =$ ____________________

c. The probability of the event C that A occur or B occurs but *not* both:

$\mathsf{P}[C] =$ ____________________

2. You choose a number n. I throw n identical fair gold coins into the air and if they all fall Heads, you get to keep them all; but if any falls Tails you get nothing. How many coins would you like me to throw? Why?

OPTIONAL (for extra credit): You choose a number n. I roll n identical gold dice and if none shows an Ace, you get to keep them all; but if any ace appears you get nothing. How many dice would you like me to roll? Why?

3. The three doors on the Monty Hall show open to show a goat; another goat; and a new car. If a goat cost \$1,000 and a car cost \$10,000, and if each contestant chooses a door at random (probability 1/3 each) and sticks with it (no "switching"),

a. What is the expected value of one contestant's prize, X?

$\mathsf{E}[X] =$ __________________

b. What is the variance?

$\mathsf{V}[X] =$ __________________

c. Let X_n be the prize won by the n^{th} contestant and let $S_{100} = X_1 + ... + X_{100}$ be the sum. In 100 shows, how much money should Monty Hall's producers expect to spend on prizes? What would be the variance? Show your work.

$\mathsf{E}[S_{100}] =$ __________________ $\mathsf{V}[S_{100}] =$ __________________

d. Approximately what is the probability that the prize total will exceed \$350,000? Show your work.

$\mathsf{P}[S_{100} \geq \$350,000] \approx$ __________________

4. Every time Chris and Pat run the 100 meter dash they do it in different times; Chris's times (in seconds) follow the exponential distribution $f(t) = \lambda e^{-\lambda t}$ (for $t > 0$) with $\lambda = 0.10$, while Pat's times follow the same distribution with $\lambda = 0.08$.

a. What is the probability that Chris beats Pat in a $100m$ dash?

$\mathsf{P}[C < P] =$ ________________

b. What is the probability that Chris beats the world record, $9.88s$?

$\mathsf{P}[C < 9.88] =$ ________________

5. Throw a fair die three times, and call the numbers thrown on the three trials X_1, X_2, and X_3.

a. What is the probability that the middle number is (strictly) larger than the other two?

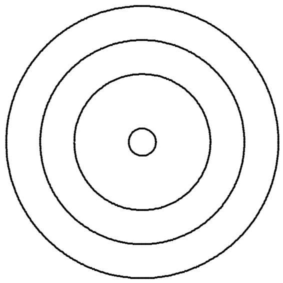
$\mathsf{P}[X_2 > \max(X_1, X_3)] =$ ________________

b. What is the expected product?

$\mathsf{E}[X_1 \cdot X_2 \cdot X_3] =$ ________________

6. The cappuccino machine at the Cambridge Inn can be adjusted to dispense *about* μ ounces (*oz*) each time, for any $0<\mu<10$, but the actual amount dispensed is variable with a standard deviation of $0.1\,oz$. If the amounts dispensed follow a normal probability distribution, how should μ be set to ensure that only one $6\,oz$ cup in a hundred overflows, on average?

7.



Pat and Chris are equally likely to shoot an arrow over our shoulder toward the target, but they are not equally talented archers. Pat's arrows hit the target at points (R_p, θ_p) (in polar coordinates) with θ_p uniformly distributed in the interval $[0, 2\pi)$ and R_p satisfying $\mathsf{P}[R_p > r] = e^{-4r^2}$ for $r > 0$, independently; for Chris's arrows the distribution of θ is the same but $\mathsf{P}[R_c > r] = e^{-r^2}$.

a. What is the probability density function for R_p, Pat's distance from the center?

$f_p(r) =$ __________________

b. If both archers shoot, what is the probability that even the worse shot is inside the $r = 1/2$ circle?

$\mathsf{P}[\max(R_p, R_c) < 1/2] =$ __________________

c. SWOOSH! Suddenly an arrow flies over our shoulder and hits the bull's eye, $R \leq 0.10$. What is the probability that it was Pat who shot the arrow, if they were equally likely to shoot?

$\mathsf{P}[\text{Pat}] =$ __________________

8. Pat is still an archer whose arrows hit polar-coordinate points (r, θ) with θ uniformly distributed in the interval $[0, 2\pi)$ and R satisfying $\mathsf{P}[R_p > r] = e^{-4r^2}$ for $r > 0$, with R and θ independent. I'm not sure where Chris went.

a. What is the JOINT probability density function for R and θ?

$f(r, \theta) =$ __________________

b. Let $X = r\cos(\theta)$ and $Y = r\sin(\theta)$ be the Cartesian coordinates of Pat's point. It's possible to show that the joint density function for X and Y is $f_{xy}(x, y) = (4/\pi)e^{-4(x^2+y^2)}$ for $-\infty < x, y < \infty$ (note that R is not restricted to the unit interval, and X and Y need not be less than one in absolute value). Are X and Y independent? Why? (NOTE: No integration is needed)

Yes No Why?

9. Pat is *still* an archer whose arrows hit points at distance R from the center of the target satisfying $\mathsf{P}[R_p > r] = e^{-4r^2}$ for $r > 0$. The target has radius one. Chris is on the way to the hospital with an arrow wound.

a. What is the probability of the event $T = \{R \leq 1\}$ that Pat hits the target at all?

$\mathsf{P}[T] =$ __________________

b. What is the (exact or approximate) probability that Pat misses the target exactly once in 100 tries?

$\mathsf{P}[\text{Exactly One Miss}] =$ __________________

10. A US deck of playing cards contains 52 cards of four "suits" (♣, ♢, ♡, ♠) and thirteen "ranks" (2,3,4,5,6,7,8,9,10,J,Q,K,A). Imagine dealing cards (just once) from a well-shuffled deck, without replacement. Let H_1 be the event "Heart (♡) on 1^{st} card", H_2 be "Heart on 2^{nd} card", etc.; similarly let D_n be the event "Diamond (♢) on n^{th} card" and A_n be the event "Ace (A) on n^{th} card".

a. Give the indicated probabilities:

$\mathsf{P}[H_1] =$ ________ $\mathsf{P}[D_1] =$ ________ $\mathsf{P}[A_1] =$ ________

b. Give the indicated probabilities:

$\mathsf{P}[H_2] =$ ________ $\mathsf{P}[H_1 \cap H_2] =$ ________ $\mathsf{P}[H_1 | H_2]$ ________

c. Give the indicated probabilities:

$\mathsf{P}[H_1 \cap D_1] =$ ________ $\mathsf{P}[H_1 \cap A_1] =$ ________ $\mathsf{P}[A_1 \cap H_2]$ ________

d. Which pairs of events are independent? Circle them.

$\{H_1, D_1\}$ $\{H_1, A_1\}$ $\{A_1, H_2\}$ $\{H_1, H_2\}$ $\{H_1, D_2\}$

STA 104
MTH 135

Normal Distribution Table

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt:$$

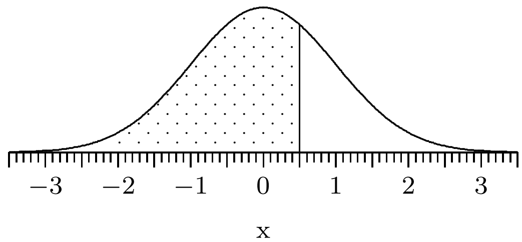


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

STA 104 Name:____________________________
MTH 135 **Probability Second Test**

Extra worksheet, if needed