Final Examination

Mth 135 = Sta 104

Monday, 1999 December 13, 2:00 – 5:00 pm

This is a closed-book examination, so please do not refer to your notes, the text, or to any other books. You may use a two-sided single sheet of *your own* notes, if you wish, but you may not share materials. A normal distribution table and a blank worksheet are attached to the exam. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other.

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. You should spend about 15 minutes on each problem. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck. After completing the exam please acknowledge the Duke Honor Code:

I have neither given nor received any unauthorized aid on this exam.

Signature: ______________________________

1.	/10
2.	/10
3.	/10
4.	/10
5.	/10
6.	/10
7.	/10
8.	/10
9.	/10
10.	/10
Total:	/100

Problem 1: Six students each roll a fair (six-sided) die, independently of course. Each of the students circles his or her number beside his or her name on the Master List:

1.	Chris	1 2 3 4 5 6
2.	Drew	1 2 3 4 5 6
3.	Lou	1 2 3 4 5 6
4.	Pat	1 2 3 4 5 6
5.	Rae	1 2 3 4 5 6
6.	Sandy	1 2 3 4 5 6

a) What is the number of possible outcomes of the entire experiment?

b) What is the probability that all six students have different numbers?

c) If the number of students who roll even numbers is at least four, what is the (conditional) probability that all six students have even numbers?

d) What is the probability that exactly one of the students rolls the number of letters in his or her name (3 for Pat, 5 for Chris, etc.)?

Problem 2: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, *Uniform* or *Weibull* and, whatever the distribution, give its mean μ:

a) The number of buses arriving in a 20 minute period, if about 6 buses arrive each hour on average:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

b) The length of time in minutes until four more buses bus arrives, if about 6 buses arrive each hour on average:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

c) The number of subjects in a continuing clinical trial until the tenth successful treatment, if the treatment is effective for about 20% of the population:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

d) The total weight of 100 Freshmen selected at random from the Class of '03, if Freshmen weigh 135 lb. on average:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

e) The number of red-swoosh Nike socks in a handful of 5 socks pulled from a drawer containing 40 socks, including eight red-swoosh Nikes:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We
$\mu =$

Problem 3: The joint probability density function for the random variables X and Y is given by

$$f(x,y) = \begin{cases} 4\,e^{-2y} & \text{for } 0 < x \le y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the marginal probability density function for X, identify it by name, and give its mean: $f_X(x) =$ ____________________________;

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We

$\mu =$

b) Find the marginal pdf for Y, identify it by name, and give its mean: $f_Y(y) =$ ____________________________;

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We

$\mu =$

c) Find the *conditional* pdf for X, given $Y = 4$; identify it by name, and give its mean: $f_{X|Y}(x \mid 4) =$ ____________________________;

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un ◯ We

$\mu =$

Problem 4: Santa's sleigh will work fine with seven operational reindeer, but he uses eight to increase reliability. His reliability requirement is that the failure probability be no more than one in a thousand, *i.e.*, one catastrophic failure per millennium.

a) Suppose that each reindeer is fully operational with the same probability p, and assume that raindeer failures are independent. How large must p be to ensure that Santa's sleigh will attain its requirement of 99.9% reliability? Circle one and explain:

$p \approx$ .75 .875 .94 .99 .994 .999 .9994

b) How might the assumption of independence fail?

Problem 5: A hat contains n coins, f of which are fair (so that $\mathsf{P}[H] = 1/2$) and b of which are biased with $\mathsf{P}[H] = 2/3$, with $n = f + b$. Be sure to **simplify** your answers below.

a) A single coin is drawn at random from the hat and tossed once. It lands Heads. What is the probability that it is a biased coin?

b) It is tossed a second time. What is the probability that it will show Heads this time too? (Give the *conditional* probability, given Heads on the first toss).

c) If it does show Heads on both the first and second tosses, what is the probability that it is a biased coin?

Problem 6: Let X be an exponentially distributed random variable with parameter $\lambda = 4$ and define the random variable Y by $Y = 1/X$.

a) $f_Y(y) =$ ___________________________
Determine the probability density function (pdf) for Y, at all real numbers y.

b) $\mathsf{P}[0.5 \leq Y \leq 4] =$ _____________
Compute the indicated probability as a decimal, correct to four places. (**Hint**: this is easy).

c) $\mathsf{E}[e^X] =$ _____________
Find the expected value of $\exp(X) = \exp(1/Y)$.

Problem 7: A random variable N is uniformly distributed on $\{1, 2, \cdots, 10\}$. Let X be the indicator of the event $(N \le 5)$ and Y the indicator of the event (N is even), *i.e.*,

$$X = \begin{cases} 1 & \text{if } N \in \{1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases} \qquad Y = \begin{cases} 1 & \text{if } N \in \{2,4,6,8,10\} \\ 0 & \text{otherwise} \end{cases}$$

a) $\mathsf{E}[X - Y] =$ ____________ $\mathsf{E}[2^{X-Y}] =$ ____________
Find the indicated expectations.

b) Are X and Y independent? Circle one and explain: Yes No

c) Find the CDF of $Z \equiv X + Y$: $F_Z(z) = \Bigg\{$

Problem 8: Let $X, Y, Z \sim \mathsf{No}(0, 1)$ be independent normally distributed random variables, each with mean 0 and variance 1. Calculate to four decimal places:

a) $\mathsf{P}[|X| < 1,\ |Y| < 2,\ |Z| < 3] =$ ____________
(the probability that all three inequalities hold, simultaneously)

b) $\mathsf{E}[(X + Y + Z)^2] =$ ____________

c) $\mathsf{P}[X + Y < 2Z + 1.08] =$ ____________

Problem 9: Suppose that on average one person in a hundred has a particular genetic defect, which can be detected only by a laboratory test. Some number N of subjects are chosen at random and tested; denote by X the number found to have the defect.

a) $\mathsf{P}[X \geq 1] =$ ____________
Suppose $N =$ fifty people chosen at random are tested. What is the probability that at least one of them will have the defect? (Give answer as a decimal to four places).

b) $N \geq$ ____________
How many people would have to be tested in order for the probability to be at least 99% that at least one person has the defect?

c) $\mathsf{E}[X] =$ ____________
If that many people are tested, what is the expected number of tested individuals with the defect?

Problem 10: Telephone calls arrive at an exchange at an average rate of one every second. Find the probabilities of the following events, explaining briefly your assumptions:

a) No calls arriving in a given five-second period.

b) Between four and six (inclusive) calls arriving in a five-second period.

c) Between 90 and 110 (inclusive) calls arriving in a 100-second period (give **approximate** answer to four decimals; extra credit if approximation is correct within ± 0.001)

d) Fourth call arrives at **exactly** time $T_4 = 5$ seconds.

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt:$$

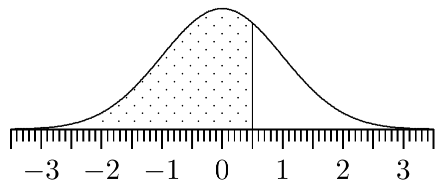

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998