STA 213

Problem 5.4

(a)

$$\begin{aligned}
\Pr(X_1 = x_1, \dots, X_k = x_k) &= \int_0^1 \left[\prod_{i=1}^k f(x_i \mid p)\right] f(p)dp \\
&= \int_0^1 p^t(1-p)^{k-t}dp \\
&= B(t+1, k-t+1)\int_0^1 \frac{1}{B(t+1,k-t+1)}p^t(1-p)^{k-t}dp \\
&= B(t+1,k-t+1) = \frac{\Gamma(t+1)\Gamma(k-t+1)}{\Gamma(k+2)} \\
&= \frac{t!(k-t)!}{(k+1)!}
\end{aligned}$$

where $t = \sum_{i=1}^k x_i$ and $B(\alpha, \beta)$ is the *beta* function. Note that $\Pr(X_1 = x_{i_1}, \dots, X_k = x_{i_k}) = \frac{t!(k-t)!}{(k+1)!}$ for any permutation $(i_1, \dots, i_k)$ of the indices $(1, \dots, k)$.

(b) From part (a) we can derive that

$$\begin{aligned}
\Pr(X_i = x_i) &= \int_0^1 f(x_i \mid p)f(p)dp = \int_0^1 p^{x_i}(1-p)^{1-x_i}dp \\
&= \frac{x_i!(1-x_i)!}{2!} = \frac{1}{2} \qquad \text{for } x_i = 0, 1
\end{aligned}$$

Therefore

$$\Pr(X_1 = x_1, \dots, X_n = x_n) = \frac{t!(n-t)!}{(n+1)!} \neq \prod_{i=1}^n \Pr(X_i = x_i) = \left(\frac{1}{2}\right)^n$$

where $t = \sum_{i=1}^{n} x_i$