STA 213

Problem 5.15

(a) For any $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\left|\sqrt{X_n} - \sqrt{a}\right| > \epsilon\right) &= \Pr\left(\left|\sqrt{X_n} - \sqrt{a}\right|\left|\sqrt{X_n} + \sqrt{a}\right| > \epsilon\left|\sqrt{X_n} + \sqrt{a}\right|\right) \\
&= \Pr\left(|X_n - a| > \epsilon\left|\sqrt{X_n} + \sqrt{a}\right|\right) \\
&\leq \Pr\left(|X_n - a| > \epsilon\sqrt{a}\right) \to 0, \\
&\quad \text{as } n \to \infty, \text{ since } X_n \to a \text{ in probability.}
\end{aligned}$$

Thus $\sqrt{X_n} \to \sqrt{a}$.

(b) For any $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\left|\frac{a}{X_n} - 1\right| \leq \epsilon\right) &= \Pr\left(\frac{a}{1+\epsilon} \leq X_n \leq \frac{a}{1-\epsilon}\right) \\
&= \Pr\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1-\epsilon}\right) \\
&\geq \Pr\left(a - \frac{a\epsilon}{1+\epsilon} \leq X_n \leq a + \frac{a\epsilon}{1+\epsilon}\right) \\
&= \Pr\left(|X_n - a| \leq \frac{a\epsilon}{1+\epsilon}\right) \to 1, \\
&\quad \text{as } n \to \infty, \text{ since } X_n \to a \text{ in probability.}
\end{aligned}$$

Thus $a/X_n \to 1$ in probability.

(c) $S_n^2 \to \sigma^2$ in probability. By (a), $S_n = \sqrt{S_n^2} \to \sqrt{\sigma^2} = \sigma$ in probability. By (b), $\sigma/S_n \to 1$ in probability.