STA 213

Problem 5.31
If $X \sim t_p$, then $X = Z/\sqrt{V/p}$ where $Z \sim N(0,1)$, $V \sim \chi^2_p$ and Z and V are independent.

(a) $E(X) = E(Z\sqrt{V/p}) - E(Z)E(1\sqrt{V/p}) = 0$ since $E(Z) = 0$, as long as the other expectation is finite. This is so if $p > 1$. Alternatively, you can simply notice that the t distribution is a unimodal distribution symmetrical about $x = 0$. All finite moments of odd order about the origin are therefore zero. Thus $E(X) = 0$. Note that $E(X)$ does not exist for $p = 1$.

From part (b), $X^2 \sim F_{1,p}$. Thus $Var(X) = E(X^2) = p/(p-2)$, if $p > 2$. Alternatively,

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{p\pi}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} x^2 \left(1 + \frac{x^2}{p}\right)^{-(p+1)/2} dx \\ &= 2\int_{0}^{\infty} \frac{1}{\sqrt{p\pi}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} x^2 \left(1 + \frac{x^2}{p}\right)^{-(p+1)/2} dx \end{aligned}$$

Set

$$y = \frac{\frac{x^2}{p}}{1 + \frac{x^2}{p}} \qquad \Longrightarrow \qquad x = \sqrt{\frac{yp}{1-y}}$$

Then

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{yp}{1-y}\right)^{-\frac{1}{2}} \frac{p}{(1-y)^2}$$

$$
\begin{aligned}
E(X^2) &= 2\int_0^1 \frac{1}{\sqrt{p\pi}}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{yp}{1-y}\left(1+\frac{1}{p}\frac{yp}{1-y}\right)^{-(p+1)/2}\frac{1}{2}\left(\frac{yp}{1-y}\right)^{-\frac{1}{2}}\frac{p}{(1-y)^2}dy \\
&= \int_0^1 \frac{1}{\sqrt{p\pi}}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\left(\frac{yp}{1-y}\right)^{1-\frac{1}{2}}\frac{p}{(1-y)^2}\left(\frac{1}{1-y}\right)^{-\frac{p+1}{2}}dy \\
&= p\sqrt{p}\frac{1}{\sqrt{p\pi}}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\int_0^1 y^{1/2}(1-y)^{\frac{p-4}{2}}dy \\
&= p\sqrt{p}\frac{1}{\sqrt{p\pi}}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{p}{2}-1\right)}{\Gamma\left(\frac{p+1}{2}\right)} \\
&= \frac{1}{2}\frac{p\sqrt{\pi}\Gamma\left(\frac{p}{2}-1\right)}{\sqrt{\pi}\Gamma\left(\frac{p}{2}\right)} \\
&= \frac{p}{2}\frac{1}{\left(\frac{p}{2}-1\right)} \\
&= \frac{p}{p-2}
\end{aligned}
$$

Note that for $p \leq 2$ the integral does not converge.

(b) $X^2 = Z^2/(V/p)$. $Z^2 \sim \chi_1^2$, so the ratio is distributed $F_{1,p}$.

(c) The pdf of X is

$$
f_X(x) = \left[\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\sqrt{p\pi}\right]\frac{1}{(1+x^2/p)^{\frac{p+1}{2}}}
$$

Denote the quantity in square brackets by C_p. From page 39 of the text book (Stirling's Formula) we have

$$
\begin{aligned}
\lim_{p\to\infty} C_p &= \lim_{p\to\infty}\frac{\sqrt{2\pi}\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}e^{-\frac{p-1}{2}}}{\sqrt{2\pi}\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}e^{-\frac{p-2}{2}}}\frac{1}{\sqrt{p\pi}} \\
&= \frac{e^{-1/2}}{\sqrt{\pi}}\lim_{p\to\infty}\frac{\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}+\frac{1}{2}}}{\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}+\frac{1}{2}}} = \frac{e^{-1/2}}{\sqrt{\pi}}\frac{e^{1/2}}{\sqrt{2}}
\end{aligned}
$$

by an application of Lemma 2.3.1 in the text book. Applying the Lemma again shows that for each x

$$\lim_{p\to\infty}(1+x^2/p)^{(p+1)/2} = e^{x^2/2}$$

establishing the result.

(d) As the random variable $F_{1,p}$ is the square of a t_p, we conjecture that it would converge to the square of a $N(0,1)$ random variable, a χ^2_1.

(e) The random variable $qF_{q,p}$ can be thought as the sum of q random variables, each a t_p squared. Thus, by all of the above, we expect it to converge to a χ^2_q random variable as $p \to \infty$.