STA 213

Problem 7.13

$L(\theta \mid x) = \prod_{i=1}^{n} \frac{1}{2} e^{-\frac{1}{2}|x_i-\theta|} = \frac{1}{2^n} e^{-\frac{1}{2}\sum |x_i-\theta|}$, so the MLE minimizes $\sum |x_i - \theta| = \sum |x_{(i)} - \theta|$, where $x_{(1)}, \ldots, x_{(n)}$ are the order statistics. For $x_{(j)} \leq \theta \leq x_{(j+1)}$,

$$\sum_{i=1}^{n} |x_{(i)} - \theta| = \sum_{i=1}^{j} (\theta - x_{(i)}) + \sum_{i=j+1}^{n} (x_{(i)} - \theta) = (2j - n)\theta - \sum_{i=1}^{j} x_{(i)} + \sum_{i=j+1}^{n} x_{(i)}$$

This is a linear function of θ that decreases for $j < (n/2)$ and increases for $j > (n/2)$. If n is odd, the likelihood is minimized at $\hat{\theta} = x_{((n+1)/2)}$. If n is even, $2j - n = 0$ if $j = n/2$. So the likelihood is constant between $x_{(n/2)}$ and $x_{(n/2)+1}$, and any value in this interval is an MLE. Usually the midpoint of this interval is taken as the MLE.