STA 213

Problem 7.23

(a)

$$f(\bar{x},\theta) = f(\bar{x} \mid \theta)f(\theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}e^{-n(\bar{x}-\theta)^2/(2\sigma^2)}\frac{1}{\sqrt{2\pi}\tau}e^{-(\theta-\mu)^2/2\tau^2}$$

(b) Factor the exponent in part (a) as

$$\frac{-n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 = -\frac{1}{2v^2}(\theta-\delta(x))^2 - \frac{1}{\tau^2+\sigma^2/n}(\bar{x}-\mu)^2$$

where $\delta(x) = (\tau^2\bar{x} + (\sigma^2/n)\mu)/(\tau^2+\sigma^2/n)$ and $v^2 = (\sigma^2\tau^2/n)/(\tau^2 + \sigma^2/n)$. Let $\mathrm{N}(a,b)$ denote the pdf of a normal distribution with mean a and variance b. The above factorization shows that

$$f(\bar{x},\theta) = \mathrm{N}(\theta,\sigma^2/\mathrm{n})) \times \mathrm{N}(\mu,\tau^2) = \mathrm{N}(\delta(\mathrm{x}),\mathrm{v}^2) \times \mathrm{N}(\mu,\tau^2+\sigma^2/\mathrm{n}))$$

where the marginal distribution of $\bar{X}$ is $\mathrm{N}(\mu,\tau^2+\sigma^2/\mathrm{n}))$ and the posterior distribution of $\theta \mid \bar{x}$ is $\mathrm{N}(\delta(\mathrm{x}),\mathrm{v}^2)$. This also completes part (c).