STA 213

Problem 7.27

First write

$$f(x_1, \ldots, x_n \mid \theta)\pi(\theta) \propto \exp\{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{|\theta|}{a}\}$$

where the exponent can be written

$$-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2 - \frac{|\theta|}{a} = -\frac{n}{2\sigma^2}\left(\theta - \delta_{\pm}(x)\right)^2 - \frac{n}{2\sigma^2}\left(\bar{x}^2 - \delta_{\pm}^2(x)\right)$$

where $\delta_{\pm}(x) = \bar{x} \pm \frac{\sigma^2}{na}$, where use the "+" if $\theta < 0$ and the "-" if $\theta > 0$. Thus, the posterior mean is

$$\begin{aligned} E(\theta \mid \mathbf{x}) &= \frac{\int_{-\infty}^{\infty} \theta e^{-\frac{n}{2\sigma^2}(\theta - \delta_{\pm}(x))^2} d\theta}{\int_{-\infty}^{\infty} e^{-\frac{n}{2\sigma^2}(\theta - \delta_{\pm}(x))^2} d\theta} \\ &= \frac{\int_{-\infty}^{0} \theta e^{-\frac{n}{2\sigma^2}(\theta - \delta_{+}(x))^2} d\theta + \int_{0}^{\infty} \theta e^{-\frac{n}{2\sigma^2}(\theta - \delta_{-}(x))^2} d\theta}{\int_{-\infty}^{0} e^{-\frac{n}{2\sigma^2}(\theta - \delta_{+}(x))^2} d\theta + \int_{0}^{\infty} e^{-\frac{n}{2\sigma^2}(\theta - \delta_{-}(x))^2} d\theta} \end{aligned}$$

$$\begin{aligned} \textbf{Denominator} &= \int_{-\infty}^{0} e^{-\frac{n}{2\sigma^2}(\theta - \delta_{+}(x))^2} d\theta + \int_{0}^{\infty} e^{-\frac{n}{2\sigma^2}(\theta - \delta_{-}(x))^2} d\theta \\ &= \sqrt{\frac{2\pi\sigma^2}{n}}\left[\Phi\left(-\frac{\delta_{+}(x)}{\sigma/\sqrt{n}}\right) + 1 - \Phi\left(-\frac{\delta_{-}(x)}{\sigma/\sqrt{n}}\right)\right] \\ &= K \end{aligned}$$

where $\Phi(\cdot)$ is the cdf of a standard normal random random variable.

$$\begin{aligned}
\textbf{Numerator} &= \int_{-\infty}^{0} \theta e^{-\frac{n}{2\sigma^2}(\theta-\delta_+(x))^2} d\theta + \int_{0}^{\infty} \theta e^{-\frac{n}{2\sigma^2}(\theta-\delta_-(x))^2} d\theta \\
&= \int_{-\infty}^{0} (\theta-\delta_+(x)) e^{-\frac{n}{2\sigma^2}(\theta-\delta_+(x))^2} d\theta + \delta_+(x) \int_{-\infty}^{0} e^{-\frac{n}{2\sigma^2}(\theta-\delta_+(x))^2} d\theta \\
&+ \int_{0}^{\infty} (\theta-\delta_-(x))\, e^{-\frac{n}{2\sigma^2}(\theta-\delta_-(x))^2} d\theta + \delta_-(x) \int_{0}^{\infty} e^{-\frac{n}{2\sigma^2}(\theta-\delta_-(x))^2} d\theta \\
&= -\frac{2\sigma^2}{n} e^{-\frac{n}{2\sigma^2}\delta_+^2(x)} + \delta_+(x)\sqrt{\frac{2\pi\sigma^2}{n}}\Phi\left(-\frac{\delta_+(x)}{\sigma/\sqrt{n}}\right) \\
&+ \frac{2\sigma^2}{n} e^{-\frac{n}{2\sigma^2}\delta_-^2(x)} + \delta_-(x)\sqrt{\frac{2\pi\sigma^2}{n}}\left[1-\Phi\left(-\frac{\delta_-(x)}{\sigma/\sqrt{n}}\right)\right] \\
&= \sqrt{\frac{2\pi\sigma^2}{n}}\left\{\delta_+(x)\Phi\left(-\frac{\delta_+(x)}{\sigma/\sqrt{n}}\right) + \delta_-(x)\left[1-\Phi\left(-\frac{\delta_-(x)}{\sigma/\sqrt{n}}\right)\right]\right\} \\
&+ \frac{2\sigma^2}{n}\left[e^{-\frac{n}{2\sigma^2}\delta_-^2(x)} - e^{-\frac{n}{2\sigma^2}\delta_+^2(x)}\right]
\end{aligned}$$

Thus

$$\begin{aligned}
E(\theta \mid \mathbf{x}) &= \frac{\sqrt{\frac{2\pi\sigma^2}{n}}\left\{\delta_+(x)\Phi\left(-\frac{\delta_+(x)}{\sigma/\sqrt{n}}\right) + \delta_-(x)\left[1-\Phi\left(-\frac{\delta_-(x)}{\sigma/\sqrt{n}}\right)\right]\right\}}{K} \\
&+ \frac{\frac{2\sigma^2}{n}\left[e^{-\frac{n}{2\sigma^2}\delta_-^2(x)} - e^{-\frac{n}{2\sigma^2}\delta_+^2(x)}\right]}{K}
\end{aligned}$$