STA 213

Problem 7.55

We will do a more general problem that includes (a) and (b) as special case. Suppose we want to estimate $\lambda e^{-\lambda t} = \Pr(X = t)$. Let $T(X_1, \ldots, X_n) = 1$, if $X_1 = t$, and $T(X_1, \ldots, X_n) = 1$, if $X_1 \neq t$. Then $E(T) = \Pr(T = 1) = \Pr(X_1 = t)$, so T is an unbiased estimator. The Poisson family is a one-parameter exponential family with $t(x) = x$, so $\sum X_i$ is a complete sufficient statistic for λ and we can apply Rao-Blackwell Theorem and find $E(T \mid \sum X_i = y)$. $T = 0$, if $y < t$, so if $y < T$, $E(T \mid \sum X_i = y) = 0$. For $y \geq t$, the estimator is

$$\begin{aligned} E(T \mid \sum X_i = y) &= \Pr(X_1 = t \mid \sum X_i = y) = \frac{\Pr(X_1 = t, \sum X_i = y)}{\Pr(\sum X_i = y)} \\ &= \frac{\Pr(X_1 = t, \sum_{i=2}^n X_i = y - t)}{\Pr(\sum X_i = y)} \\ &= \frac{\Pr(X_1 = t) \Pr(\sum_{i=2}^n X_i = y - t)}{\Pr(\sum X_i = y)} \\ &= \frac{(\lambda^t e^{-\lambda}/t!)([(n-1)\lambda]^{y-t} e^{-(n-1)\lambda}/(y-t)!)}{(n\lambda)^y e^{-n\lambda}/y!} \\ &= \binom{y}{t} \frac{(n-1)^{y-t}}{n^y} \end{aligned}$$

Thus, the estimator for $e^{-\lambda}$ is $[(n-1)/n]^y$ and the estimator for $\lambda e^{-\lambda}$ is $[y/(n-1)][(n-1)/n]^y$

(c) For these data, n=15, $\sum X_i = y = 104$ and the MLE of λ is $\hat{\lambda} = \bar{X} = 6.9333$. For $t = 0$, the estimates of $\Pr(X = 0)$ are

UNBIASED $= .000765$ MLE: $e^{-6.9333} = .000975$

For $t = 1$, the estimates of $\Pr(X = 1)$ are

UNBIASED $= .005684$ MLE: $6.9333e^{-6.9333} = .006758$