# STA 213

**Problem 8.6**

(a)

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y}) &= \frac{\sup_{\Theta_0} L(\theta \mid \mathbf{x},\mathbf{y})}{\sup_{\Theta} L(\theta \mid \mathbf{x},\mathbf{y})} = \frac{\sup_\theta \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} \prod_{i=1}^m \frac{1}{\theta} e^{-y_i/\theta}}{\sup_{\theta,\mu} \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} \prod_{i=1}^m \frac{1}{\mu} e^{-y_i/\theta}} \\
&= \frac{\sup_\theta \frac{1}{\theta^{m+n}} e^{-\left(\sum_{i=1}^n x_i + \sum_{i=1}^m y_i\right)/\theta}}{\sup_{\theta,\mu} \frac{1}{\theta^n} e^{-\left(\sum_{i=1}^n x_i/\theta\right)} \frac{1}{\mu^m} e^{-\left(\sum_{i=1}^m y_i\right)/\mu}}
\end{aligned}$$

Differantion will show that in the numerator $\hat{\theta}_0 = \left(\sum_{i=1}^n x_i + \sum_{i=1}^m y_i\right)/(n+m)$, while in the denominator $\hat{\theta} = \bar{x}$ and $\hat{\mu} = \bar{y}$. Therefore,

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y}) &= \frac{\left(\frac{n+m}{\sum_{i=1}^n x_i + \sum_{i=1}^m y_i}\right)^{n+m} e^{-\left(\frac{n+m}{\sum_{i=1}^n x_i + \sum_{i=1}^m y_i}\right)\left(\sum_{i=1}^n x_i + \sum_{i=1}^m y_i\right)}}{\left(\frac{n}{\sum_{i=1}^n x_i}\right)^n e^{-\left(\frac{n}{\sum_{i=1}^n x_i}\right)\sum_{i=1}^n x_i} \left(\frac{m}{\sum_{i=1}^m y_i}\right)^m e^{-\left(\frac{m}{\sum_{i=1}^m y_i}\right)\sum_{i=1}^m y_i}} \\
&= \frac{(m+n)^{n+m}}{n^n m^m} \frac{\left(\sum x_i\right)^n \left(\sum y_i\right)^m}{\left(\sum x_i + \sum y_i\right)^{n+m}}
\end{aligned}$$

And the LRT is to reject $H_0$ if $\lambda(\mathbf{x},\mathbf{y}) \le c$.

(b)

$$\lambda = \frac{(m+n)^{n+m}}{n^n m^m} \left(\frac{\sum x_i}{\sum x_i + \sum y_i}\right)^n \left(\frac{\sum y_i}{\sum x_i + \sum y_i}\right)^m = \frac{(m+n)^{n+m}}{n^n m^m} T^n (1-T)^m$$

Therefore $\lambda$ is a function of $T$. $\lambda$ is a unimodal function of $T$ which is maximized when $T = \frac{n}{m+n}$. Rejection for $\lambda \le c$ is equivalent to rejection for $T \le a$ or $T \ge b$, where $a$ and $b$ are constants that satisfy $a^n(1-a)^m = b^n(1-b)^m$.

(c) When $H_0$ is true, $\sum X_i \sim$ gamma$(n, \theta)$ and $\sum Y_i \sim$ gamma$(m, \theta)$ and they are independent. So $T \sim$ beta$(n, m)$.