STA 213

Problem 8.7

(a)

$$L(\theta, \lambda \mid \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\lambda} e^{-(x_i-\theta)/\lambda} I_{[\theta,\infty)}(x_i) = \left(\frac{1}{\lambda}\right)^n e^{-(\sum x_i - n\theta)/\lambda} I_{[\theta,\infty)}(x_{(1)})$$

which is increasing in θ if $x_{(1)} \geq \theta$ (regardless of λ). So the MLE of θ is $\hat{\theta} = x_{(1)}$. Then

$$\frac{\partial \log L}{\partial \lambda} = -\frac{n}{\lambda} + \frac{\sum x_i - n\hat{\theta}}{\lambda^2} = 0 \Rightarrow n\hat{\lambda} = \sum x_i - n\hat{\theta} \Rightarrow \hat{\lambda} = \bar{x} - x_{(1)}$$

Since

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{n}{\lambda^2} - 2\left.\frac{\sum x_i - n\hat{\theta}}{\lambda^3}\right|_{\bar{x}-x_{(1)}} = \frac{n}{(\bar{x}-x_{(1)})^2} - \frac{2n(\bar{x}-x_{(1)})}{(\bar{x}-x_{(1)})^3} = \frac{-n}{(\bar{x}-x_{(1)})^2} < 0$$

we have $\hat{\theta} = x_{(1)}$ and $\hat{\lambda} = \bar{x} - x_{(1)}$ as the unrestricted MLEs of θ and λ. Under the restriction $\theta \leq 0$, the MLE of θ (regardless of λ) is

$$\hat{\theta}_0 = \begin{cases} 0 & \text{if } x_{(1)} > 0 \\ x_{(1)} & \text{if } x_{(1)} \leq 0 \end{cases}$$

For $x_{(1)} > 0$, substituting $\hat{\theta}_0 = 0$ and maximizing with respect to λ , as above, yields $\hat{\lambda} = \bar{x}$. Therefore,

$$\begin{aligned} \lambda(\mathbf{x}) &= \frac{\sup_{\Theta_0} L(\theta, \lambda \mid \mathbf{x})}{\sup_{\Theta} L(\theta, \lambda \mid \mathbf{x})} = \frac{\sup_{\{(\lambda,\theta):\theta\leq 0\}} L(\theta, \lambda \mid \mathbf{x})}{L(\hat{\theta}, \hat{\lambda} \mid \mathbf{x})} \\ &= \begin{cases} 1, & \text{if } x_{(1)} > 0 \\ \frac{L(\bar{x},0|\mathbf{x})}{L(\hat{\theta},\hat{\lambda}|\mathbf{x})} & \text{if } x_{(1)} \leq 0 \end{cases} \end{aligned}$$

where

$$\frac{L(\bar{x}, 0 \mid \mathbf{x})}{L(\hat{\theta}, \hat{\lambda} \mid \mathbf{x})} = \frac{(1/\bar{x})^n e^{-n\bar{x}/\bar{x}}}{(1/\hat{\lambda})^n e^{-n(\bar{x}-x_{(1)})/(\bar{x}-x_{(1)})}} = \left(\frac{\hat{\lambda}}{\bar{x}}\right)^n = \left(\frac{\bar{x}-x_{(1)}}{\bar{x}}\right)^n = \left(1 = \frac{x_{(1)}}{\bar{x}}\right)^n$$

So rejecting if $\lambda(\mathbf{x}) \le c$ is equivalent to rejecting if $x_{(1)}/\bar{x} \ge c^\star$, where $c^\star$ is some constant.

(b) The LRT statistic is

$$\lambda(\mathbf{x}) = \frac{\sup_\beta (1/\beta^n) e^{-\sum x_i/\beta}}{\sup_{\beta,\gamma} (\gamma^n/\beta^n) \left(\prod x_i\right)^{\gamma-1} e^{-\sum x_i^\gamma/\beta}}$$

The numerator is maximized at $\hat{\beta}_0 = \bar{x}$. For fixed γ, the denominator is maximized at $\hat{\beta}_\gamma = \sum x_i^\gamma/n$. Thus

$$\lambda(\mathbf{x}) = \frac{\bar{x}^{-n}}{\sup_\gamma (\gamma^n/\hat{\beta}_\gamma^n) \left(\prod x_i\right)^{\gamma-1}}$$

The denominator cannot be maximized in closed form. Numeric maximization could be used to compute the statistic for observed data $\mathbf{x}$.