# STA 213

**Problem 3.24**

(a) $f_X(x) = \frac{1}{\beta}\exp\{-\frac{x}{\beta}\}$, $x > 0$. For $Y = X^{1/\gamma}$, $f_Y(y) = \frac{\gamma}{\beta}e^{-y^\gamma/\beta}y^{\gamma-1}$, $y > 0$. Using the transformation $z = y^\gamma/\beta$, we calculate

$$E(Y^n) = \frac{\gamma}{\beta}\int_0^\infty e^{-y^\gamma/\beta}y^{\gamma+n-1}dy = \beta^{n/\gamma}\int_0^\infty z^{n/\gamma}e^{-z}dz = \beta^{n/\gamma}\Gamma\left(\frac{n}{\gamma}+1\right)$$

Thus $E(Y) = \beta^{1/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right)$ and $Var(Y) = \beta^{2/\gamma}\left[\Gamma\left(\frac{2}{\gamma}+1\right) - \Gamma^2\left(\frac{1}{\gamma}+1\right)\right]$.

(b) $f_X(x) = \frac{1}{\beta}\exp\{-\frac{x}{\beta}\}$, $x > 0$. For $Y = (2X/\beta)^{1/2}$, $f_Y(y) = ye^{-y^2/2}$, $y > 0$. We now notice that

$$E(Y) = \int_0^\infty y^2 e^{-y^2/2}dy = \frac{\sqrt{2\pi}}{2}$$

since $(1/\sqrt{2\pi})\int_{-\infty}^\infty y^2e^{-y^2/2}dy = 1$, the variance of a standard normal, and the integrand is symmetric. Use integration by parts to calculate the second moment

$$E(Y^2) = \int_0^\infty y^3 e^{-y^2/2}dy = 2\int_0^\infty ye^{-y^2/2}dy = 2$$

where we take $u = y^2$, $dv = ye^{-y^2/2}$. Thus $Var(Y) = 2(1 - \pi/4)$.

(c) $f_X(x) = \frac{1}{\Gamma(3/2)\beta^{3/2}}x^{3/2-1}e^{-x/\beta}$, $x > 0$. For $Y = (X/\beta)^{1/2}$, $f_Y(y) = \frac{2}{\Gamma(3/2)}y^2e^{-y^2}$, $y > 0$. To calculate the moments we use integration by parts.

$$
\begin{aligned}
E(Y) &= \frac{2}{\Gamma(3/2)}\int_0^\infty y^3 e^{-y^2}dy = \frac{2}{\Gamma(3/2)}\int_0^\infty y e^{-y^2}dy = \frac{1}{\Gamma(3/2)} \qquad (u=y^2, dv = ye^{-y^2}) \\
E(Y^2) &= \frac{2}{\Gamma(3/2)}\int_0^\infty y^4 e^{-y^2}dy = \frac{3}{\Gamma(3/2)}\int_0^\infty y^2 e^{-y^2} \\
&= \frac{3\sqrt{\pi}}{4\Gamma(3/2)} = \frac{3}{2} \qquad (u=y^3, dv = ye^{-y^2})
\end{aligned}
$$

Using the fact that $(1/\sqrt{\pi})\int_{-\infty}^\infty y^2 e^{-y^2}dy = \frac{1}{2}$, since it is the variance of a N(0,1/2), symmetry yields $\int_0^\infty y^2 e^{-y^2}dy = \frac{\sqrt{\pi}}{4}$. Thus, $Var(Y) = \frac{3}{2} - \frac{4}{\pi}$, using $\Gamma(3/2) = \sqrt{\pi}/2$.

(d) $f_X(x) = e^{-x}$, $x > 0$. For $Y = \alpha - \gamma\log X$, $f_Y(y) = \frac{1}{\gamma}e^{(\alpha-y)/\gamma}e^{-e^{(\alpha-y)/\gamma}}$, $-\infty < y < \infty$. Calculation of $E(Y)$ and $E(Y^2)$ cannot be done in closed form. If we define

$$
I_1 = \int_0^\infty (\log x)e^{-x}dx, \qquad I_2 = \int_0^\infty (\log x)^2 e^{-x}dx
$$

then

$$
\begin{aligned}
E(Y) &= E(\alpha - \gamma\log X) = \alpha - \gamma I_1 \\
E(Y^2) &= E[(\alpha-\gamma\log X)^2] = \alpha^2 - 2\alpha\gamma I_1 + \gamma^2 I_2
\end{aligned}
$$

The constant $I_1 = 0.5772157$ is called Euler's constant.