STA 213

Problem 4.

(a)

$$\Pr(Z \le z, W = 0) = \Pr(\min(X, Y) \le z, Y \le X) = \Pr(Y \le z, Y \le X) =$$

$$= \int_0^z \int_y^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy = \frac{\lambda}{\mu + \lambda} \left(1 - \exp\left\{-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z\right\}\right)$$

Similarly,

$$\Pr(Z \le z, W = 1) = \Pr(\min(X, Y) \le z, X \le Y) = \Pr(X \le z, X \le Y) =$$

$$= \int_0^z \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy = \frac{\mu}{\mu + \lambda} \left(1 - \exp\left\{-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z\right\}\right)$$

(b)

$$\Pr(W = 0) = \Pr(Y \le X) = \int_0^\infty \int_y^\infty \frac{1}{\lambda} e^{-x/\lambda} \frac{1}{\mu} e^{-y/\mu} dx dy = \frac{\lambda}{\mu + \lambda}$$

$$\Pr(W = 1) = 1 - \Pr(W = 0) = \frac{\mu}{\mu + \lambda}$$

$$\Pr(Z \le z) = \Pr(Z \le z, W = 0) + \Pr(Z \le z, W = 1) = 1 - \exp\left\{-\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) z\right\}$$

Therefore, $\Pr(Z \le z \mid W = i) = \Pr(Z \le z) \Pr(W = i)$, for $i = 0, 1, z > 0$. So Z and W are independent.