STA 213

Problem 4.42

(a)

$$\begin{aligned}
E(X) &= a_X E(Z_1) + b_X E(Z_2) + E(c_X) = a_X 0 + b_X 0 + c_X = c_X \\
Var(X) &= a_X^2 Var(Z_1) + b_X^2 Var(Z_2) + Var(c_X) = a_X^2 + b_X^2 \\
E(Y) &= a_Y 0 + b_Y 0 + c_Y = c_Y \\
Var(X) &= a_Y^2 Var(Z_1) + b_Y^2 Var(Z_2) + Var(c_Y) = a_Y^2 + b_Y^2 \\
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= E[(a_X a_Y Z_1^2 + b_X b_Y Z_2^2 + c_X c_Y + a_X b_Y Z_1 Z_2 + a_X c_Y Z_1 \\
&\quad + b_X a_Y Z_2 Z_1 + b_X c_Y Z_2 + c_X a_Y Z_1 + c_X b_Y Z_2) - c_X c_y] \\
&= a_X a_Y + b_X b_Y
\end{aligned}$$

since $E(Z_1^2) = E(Z_2^2) = 1$ and the expectations of other terms are all zero.

(b) Simply plug the expressions for a_X, b_x, etc. into the equalities in (a) and simplify.

(c) Let $D = a_X b_Y - a_Y b_X = -\sqrt{1-\rho^2}\sigma_X \sigma_Y$ and solve for Z_1 and Z_2,

$$\begin{aligned}
Z_1 &= \frac{b_Y(X - c_X) - b_X(Y - c_Y)}{D} = \frac{\sigma_Y(X - \mu_X) + \sigma_X(Y - \mu_Y)}{\sqrt{2(1+\rho)}\sigma_X \sigma_Y} \\
Z_2 &= \frac{\sigma_Y(X - \mu_X) + \sigma_X(Y - \mu_Y)}{\sqrt{2(1-\rho)}\sigma_X \sigma_Y}
\end{aligned}$$

Thus the Jacobian is

$$
\begin{aligned}
J &= \begin{bmatrix} \frac{\partial z_1}{x} & \frac{\partial z_1}{y} \\ \frac{\partial z_2}{x} & \frac{\partial z_2}{y} \end{bmatrix} = \begin{bmatrix} \frac{b_Y}{D} & \frac{-b_X}{D} \\ \frac{-a_Y}{D} & \frac{a_X}{D} \end{bmatrix} \\
&= \frac{a_X b_Y}{D^2} - \frac{a_Y b_X}{D^2} = \frac{1}{D} = \frac{1}{-\sqrt{1-\rho^2}\sigma_X \sigma_Y}
\end{aligned}
$$

and we have

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \frac{(\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y))^2}{2(1+\rho)\sigma_X^2\sigma_Y^2} \right\} \\
&\times \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \frac{(\sigma_Y(x-\mu_X)+\sigma_X(y-\mu_Y))^2}{2(1-\rho)\sigma_X^2\sigma_Y^2} \right\} \\
&\times \frac{1}{\sqrt{1-\rho^2}\sigma_X\sigma_Y} \\
&= \left(2\pi\sqrt{1-\rho^2}\sigma_X\sigma_Y\right)^{-1} \\
&\times \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right] \right\} \\
&\quad -\infty < x < \infty, \qquad -\infty < y < \infty
\end{aligned}
$$

a bivariate normal pdf.

(d) Another solution is

$$
\begin{array}{cc}
a_X = \rho\sigma_X & b_X = \sqrt{(1-\rho^2)}\sigma_X \\
a_Y = \sigma_Y & b_Y = 0 \\
c_X = \mu_X & c_Y = \mu_Y
\end{array}
$$

There are an infinite number of solutions. Write $b_X = \pm\sqrt{\sigma_X^2 - a_X^2}$, $b_Y = \pm\sqrt{\sigma_Y^2 - a_Y^2}$ and substitute b_X, b_Y into $a_X a_Y = \rho\sigma_X\sigma_Y$. We get $a_X a_Y + \left(\pm\sqrt{\sigma_X^2 - a_X^2}\right)\left(\pm\sqrt{\sigma_Y^2 - a_Y^2}\right) = \rho\sigma_X\sigma_Y$. Square both sides and simplify to get $(1-\rho^2)\,\sigma_X^2\sigma_Y^2 = \sigma_X^2 a_Y^2 - 2\rho\sigma_X\sigma_Y a_X a_Y + \sigma_Y^2 a_X^2$. This is an ellipse for $\rho \neq \pm 1$, a line for $\rho = \pm 1$. In either case there are an infinite number of points satisfying the equations.