Final Examination

Mth 135 = Sta 104

Thursday, 1997 December 18, 2:00 – 5:00 pm

This is a closed-book examination, so please do not refer to your notes, the text, or to any other books. You may use a two-sided single sheet of *your own* notes, if you wish, but you may not share materials. A normal distribution table and a blank worksheet are attached to the exam. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other.

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. Please give all numerical answers as fractions **in lowest terms** or as decimals to **four places**. You should spend about 15 minutes on each problem. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck. Please acknowledge the Duke Honor Code:

I have neither given nor received any unauthorized aid on this exam.

Signature: ______________________________

1.	/10
2.	/10
3.	/10
4.	/10
5.	/10
6.	/10
7.	/10
8.	/10
9.	/10
10.	/10
Total:	/100

Problem 1: A student takes a multiple choice exam where each question has six possible answers. He works a question correctly if he knows the answer, otherwise he guesses at random. Suppose that he knows the answers to 70% of all questions like those on the exam.

a) What is the probability that the student gets the correct answer to Problem 4?

b) Given that the student gets the correct answer to Problem 4, what is the probability that he actually knows the answer?

c) Suppose there are 20 questions on the exam. Let N be the random variable describing the number of questions the student gets correct. Find the expectation $\mathrm{E}[N]$ and variance $\mathrm{Var}[N]$.

Problem 2: The joint probability density function for the random variables X and Y is given by

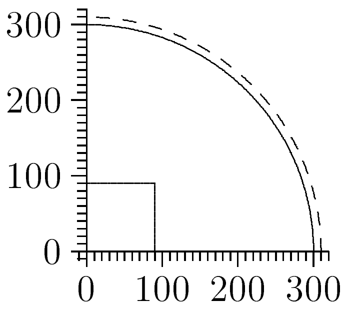
$$f(x,y) = \begin{cases} 8xy & \text{for } 0 < x \leq y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Calculate the covariance $\mathrm{Cov}[X, Y]$ of X and Y.

b) Calculate the variance $\mathrm{Var}[X + Y]$.

c) Show that X and Y are *not* independent.

Problem 3: A certain baseball field is shaped like a quarter-circle of radius 300′ (the distance from home plate to the outfield fence, in feet). Whenever Bubba bats, the ball hits the ground first at a point distributed uniformly in a quarter-circle of radius 310′.



a) What is the probability that Bubba hits the ball into the infield, a square of dimensions $90' \times 90'$?

b) What is the probability that Bubba hits a home run over the outfield fence?

c) What is the (approximate) probability that Bubba hits at least 20 home runs this season in his 400 times at bat?

Problem 4: Choose the best probability distribution for each random variable below from among the choices *Binomial*, *Geometric*, *Hypergeometric*, *Poisson*, *Uniform*, *Exponential*, or *Normal* and, whatever the distribution, give its mean μ:

a) The number of subjects in a 10-subject clinical trial who are cured by an experimental treatment, if about 20% of the population will be cured:

○ Bin ○ Geo ○ HypGeo ○ Poi ○ Uni ○ Exp ○ Norm

$\mu =$

b) The number of subjects entering a sequential clinical trial before the first one experiences an adverse reaction, if about 1% of the population does:

○ Bin ○ Geo ○ HypGeo ○ Poi ○ Uni ○ Exp ○ Norm

$\mu =$

c) The total weight of a catch of 108 fish from a lake whose average fish weighs 12oz:

○ Bin ○ Geo ○ HypGeo ○ Poi ○ Uni ○ Exp ○ Norm

$\mu =$

d) The round-off error in reporting temperatures to the nearest degree instead of the nearest .001 degree as reported by a laboratory thermometer:

○ Bin ○ Geo ○ HypGeo ○ Poi ○ Uni ○ Exp ○ Norm

$\mu =$

e) The number of men who get sick from bad clams at a dinner banquet, if 40 people attend the banquet (half men, half women) and 8 people order the bad clams:

○ Bin ○ Geo ○ HypGeo ○ Poi ○ Uni ○ Exp ○ Norm

$\mu =$

Problem 5: A hat contains \$34 in ten US bills: four one-dollar bills and six five-dollar bills. Three bills are drawn at random from the hat, without replacement. Let X denote the denomination of the smallest bill drawn and Y that of the largest bill drawn; for example, if the three bills were $1, 5, 1$ then $X = 1$, $Y = 5$.

a) Find the joint probability mass function $p(x, y)$ for X and Y.

b) Find the marginal probability mass functions $p_X(x)$, $p_Y(y)$.

c) Find the conditional probability mass function $p_Y(y|X = 1)$ of Y, given $X = 1$.

d) Are X and Y independent? Why?

Problem 6: The random variables X and Y are independent, each normally distributed, with means and variances $\mathrm{E}[X] = 3$ and $\mathrm{Var}[X] = 9$, $\mathrm{E}[Y] = 4$ and $\mathrm{Var}[Y] = 16$.

a) Give the distribution of $Z = X - Y$.

b) Find the probability $\mathrm{P}[X < 4]$.

c) Find the probability $\mathrm{P}[X < Y]$.

Problem 7: Let X be an exponentially distributed random variable with parameter $\lambda = 1$ and define the random variable Y by

$$Y = \begin{cases} X & \text{if } X \leq 1 \\ 1/X & \text{if } X > 1. \end{cases}$$

a) Find the cumulative distribution function (CDF) $F_Y(y)$, for all real numbers y.

b) Determine the probability density function (PDF) $f_Y(y)$, for all real numbers y.

c) Compute the probability $\mathrm{P}[0.5 \leq Y \leq 1.5]$.

Problem 8: Suppose that customers enter a Post Office at a Poisson rate of about one-half customer per minute, so the number of customers who arrive in any t-minute period will have a Poisson distribution with mean $t/2$.

a) Find the probability that no customer arrives in the first five minutes.

b) Let T be the length of time in minutes until the next customer arrives; find $\mathrm{P}[T \le t]$ for each $t > 0$ and give the mean $\mu = \mathrm{E}[T]$ and variance $\sigma^2 = \mathrm{Var}[T]$ of T.

c) Find the approximate probability that the 250^{th} customer of the day arrives during the last hour of operation, if the Post Office is open for eight hours a day.

Problem 9: The random variable X has mean $\mathrm{E}[X] = 1$ and variance $\mathrm{Var}[X] = 1$.

a) Find $\mathrm{P}[X \geq 2]$ if X has an exponential distribution.

b) Find $\mathrm{P}[X \geq 2]$ if X has a Poisson distribution.

c) Find $\mathrm{P}[X \geq 2]$ if X has a normal distribution.

d) Find $\mathrm{P}[X \geq 2]$ if X has a uniform distribution.

e) Find an upper bound for $\mathrm{P}[X \geq 4]$, if nothing else is known about the distribution of X but the mean and variance.

Problem 10: Karl passes three traffic signals on his way to work each day; about one day in 5 they are all green on his morning drive.

a) How many times in the 225 work days in 1997 should he expect to find all the lights green?

b) Give an expression for the exact probability that he will find all the lights green on exactly 50 work days in 1997. Do not evaluate it numerically.

c) Use the normal approximation to find the approximate probability that he will find all the lights green on exactly 50 work days in 1997. Give your answer to at least three decimal places.

Extra worksheet, if needed: