STA 104 Name:____________________________
MTH 135

Probability Midterm Exam

2:15-3:30 pm Thursday, 21 October 1999

You may use a calculator and your own notes but may not consult your books or neighbors. Please show your work for partial credit, and circle your answers. Points are awarded for *solutions*, not *answers*, so correct answers without justification will not receive full credit. Please give all numerical answers as decimals to four places. When you have finished, please sign the Duke Honor Code pledge.

I have neither given nor received aid on this examination.

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STA 104
MTH 135

Probability Midterm Exam

1. Space shuttle engines only last a few flights before they must be replaced, due to wear and sometimes damage. Denote by N the (random) number of flights a particular engine will take, and suppose that the probability mass function (pmf) for N is given by the following table:

n	1	2	3	4	5
$\mathsf{P}[N = n]$	.40	.20	.10	.20	.10

Be sure to show your work below.

a. What is the probability that an engine lasts at least three flights?

$\mathsf{P} =$ ______________

b. What is the expected engine life?

$\mathsf{E}[N] =$ ______________

c. What is the variance of the engine life?

$\mathsf{V}[N] =$ ______________

d. Which is more likely to last at least three more flights, a new engine or one that has already survived one flight? Why?

New or old? ______________

2. Consider two events A and B such that $\mathsf{P}(A) = 0.3$ and $\mathsf{P}(B) = 0.4$. Find the conditional probability of A given B under each of the following conditions:

a. A and B are independent:

$\mathsf{P}(A \mid B) =$ ______________

b. A and B are exclusive (or disjoint... so $AB = \emptyset$):

$\mathsf{P}(A \mid B) =$ ______________

c. A implies B:

$\mathsf{P}(A \mid B) =$ ______________

d. $\mathsf{P}(AB) = 0.10$:

$\mathsf{P}(A \mid B) =$ ______________

e. $\mathsf{P}(A \cup B) = 0.50$:

$\mathsf{P}(A \mid B) =$ ______________

Probability Midterm Exam

3. The Gamma distribution was introduced in class as the waiting time for the r^{th} event (catching fish, in class) when events arrive independently in disjoint time intervals, at a constant overall rate (λ fish per hour, in class). It is often used to model survival times (time-to-failure). Let T be the survival time in hours for a transistor in a high-radiation environment, and assume that T has probability density function $f(t) = 0.25te^{-0.5t}$ for $t > 0$.

a. Find the indicated probability (Hint: this can be done in at least two ways: either by integrating by parts, or (easier!) by using the relationship of the Gamma and Poisson distributions):

$\mathsf{P}[T > 4] =$ ______________

b. Find the expectation of $g(T)$ for the function $g(t) = t$. (same Hint)

$\mathsf{E}[T] =$ ______________

c. Evaluate the expectation of $g(T)$ for the function $g(t) = 1/t$.

$\mathsf{E}[1/T] =$ ______________

Probability Midterm Exam

4. The *Beta distribution* is often used to model uncertain fractions, probabilities, or proportions; Beta random variables take values only between 0 and 1, with probability density function

$$f(x) = \begin{cases} cx^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & x \le 0,\ x \ge 1 \end{cases}$$

for some constants $\alpha > 0$, $\beta > 0$ that will depend on the particular application (the Uniform Distribution is the special case where $\alpha = \beta = 1$). The constant can be shown to be $c = \Gamma(\alpha+\beta)/\Gamma(\alpha)\Gamma(\beta)$, the mean $\mu = \mathsf{E}[X] = \alpha/(\alpha+\beta)$, and the variance $\sigma^2 = \mathsf{V}[X] = \alpha\beta/(\alpha+\beta)^2(\alpha+\beta+1)$. For $\alpha = \beta = 2$ this would give $c = 6$, $\mu = 1/2$, and $\sigma^2 = 1/12$. Recall that $\Gamma(\alpha) = (\alpha-1)! = \int x^{\alpha-1}e^{-x}\,dx$.

a. For $\alpha = 2$, $\beta = 2$ find (exactly or to four correct decimals) the probability

$\mathsf{P}[1/2 < X \le 3/4] =$ ______________

b. For $\alpha = 2$, $\beta = 2$ find the conditional probability

$\mathsf{P}[X > 1/2 \mid X \le 3/4] =$ ______________

c. Define a random variable Y by $Y \equiv \frac{X}{1-X} = (1-X)^{-1} - 1$ (this is called the "odds ratio" when X is a probability or proportion). For arbitrary α and β, find the probability density function for Y. Be sure to give the pdf correctly for all real numbers y.

$f_Y(y) =$ ______________

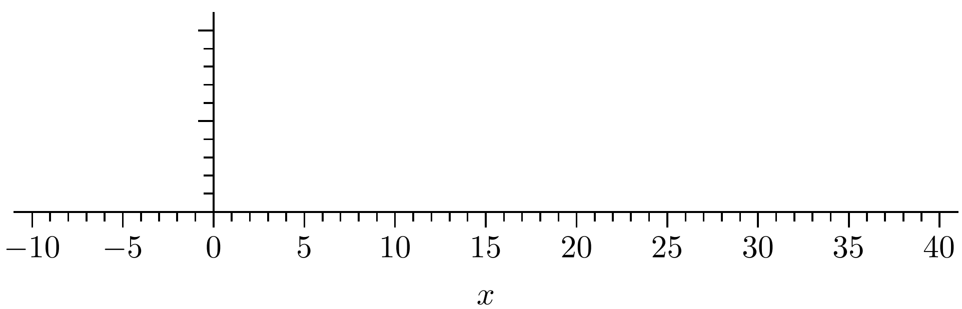
5. The probability density function of the lifetime of a certain type of artificial heart valve (measured in months) is given by

$$f(x) = \begin{cases} c/x^3 & \text{for } x > 10 \\ 0 & \text{for } x \le 10. \end{cases}$$

a. Find the value of the constant: $c =$ ______________

b. Graph the cumulative distribution function (**CDF**, not pdf) using your value of c from above. Label the vertical axis appropriately, and give the indicated values.

F(0)=__________ F(10)=__________ F(20)=__________



c. Find the probability that a valve will last between 8 and 20 months. Indicate this probability graphically in your plot in part b).

$\mathsf{P}[8 < X \le 20] =$ ______________

d. Find the hazard function for the failure time X. Does it increase, decrease, both, or stay constant for $x > 10$?

$\lambda(x) =$ ______________

Extra worksheet, if needed

STA 104
MTH 135

Normal Distribution Table

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt:$$

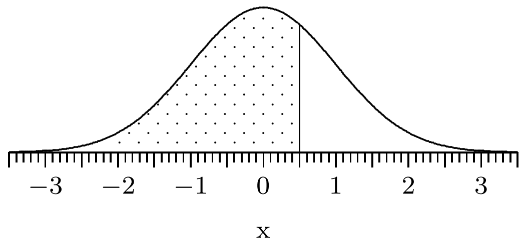


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998