STA104: solutions to Homework 3

September 28, 2001

1 Ex 7 p.173 (4 points)

Depending on how you interpret the question being asked, there are two possible ways to think about parts (a) and (b), both of which received full credit. (1 point each)
a) {6} if you consider the maximum possible value for each roll, or {1,2,3,4,5,6} if you consider all possible values that the maximum of the two rolls (individually) can be. This question is somewhat confusingly worded, but they are in fact looking for the former.
b) {1} or {1,2,3,4,5,6} similar to part (a)
c) {2,3,4,5,6,7,8,9,10,11,12}
d) {-5,-4,-3,-2,-1,0,1,2,3,4,5}

2 Ex 13 p.174 (5 points)

$$Y\text{: variable } \textit{amount of sales at first appointment} \begin{cases} 0 & \text{with prob } 0.7 \\ 500 & \text{with prob } 0.15 \\ 1000 & \text{with prob } 0.15 \end{cases}$$

$$Z\text{: variable } \textit{amount of sales at second appointment} \begin{cases} 0 & \text{with prob } 0.4 \\ 500 & \text{with prob } 0.3 \\ 1000 & \text{with prob } 0.3 \end{cases}$$

$$X\text{: variable } \textit{total amount of sales} \begin{cases} 0 & \text{with prob } [0.7\times0.4]=0.28 \\ 500 & \text{with prob } [0.7\times0.3+0.15\times0.4]=0.27 \\ 1000 & \text{with prob } [0.15\times0.4+0.3\times0.7+0.15\times0.3]=0.315 \\ 1500 & \text{with prob } [0.15\times0.3+0.15\times0.3]=0.09 \\ 2000 & \text{with prob } [0.15\times0.3]=0.045 \end{cases}$$

3 Ex 17 p.175 (4 points)

(a)

$$\begin{array}{lll} P[X=1] & = \frac{1}{2} - \frac{1}{4} & = 1/4 \\ P[X=2] & = \frac{11}{12} - \frac{3}{4} & = 1/6 \\ P[X=3] & = 1 - \frac{11}{12} & = 1/12 \end{array}$$

(b) $P[\frac{1}{2} < X < \frac{3}{2}] = \frac{5}{8} - \frac{1}{8} = 1/2$

4 Ex 30 p.178 (5 points)

St. Petersburg Paradox!
Imagine playing this game repeatedly; let $X_k =$ **player's winnings in** k^{th} **game**. Then the r.v. X_k has a discrete distribution with probability mass function:

$$P[X_k = 2^n] = 2^{-n}, \qquad n = 1, 2, ...$$

with expectation

$$E[X_k] = \sum_{n=1}^{\infty} 2^n \times 2^{-n} = 1 + 1 + 1 + 1 + ... = \infty$$

so

$$\lim_{k \to \infty} \frac{X_1 + X_2 + ... + X_k}{k} = \infty.$$

(a) Of course not; with very high probability we will lose most of our million dollars. To profit at all we would need at least nineteen Heads in a row; the probability of this is one in 524288.

(b) Well... in principle we could for-sure win arbitrarily much money by taking this bet since sooner-or-later we're bound to have a string of far more than 20 Heads. But do you *really* think your opponent can settle this bet, if you have a very long string of Heads? For example, based on today's stock prices, Bill Gates' net worth is about \$33.19B, so even Bill could not pay off if Tails appears first on or after the 32nd toss, since $\$2^{32} = \$4.3B$. This makes the *real* value of the game no more than $\sum_{n=1}^{32} f2^n(\frac{1}{2})^n = \$32$, playing against Bill, and far less playing against most other opponents—in particular, no where near \$1M.

5 Ex 32 p.178 (4 points)

Define T as the *number of tests needed for a group of ten people.* Then:

$$T = \begin{cases} 1 & \text{with probability} = (0.9)^{10} \\ 11 & \text{with probability} = 1 - (0.9)^{10} \end{cases}$$

It follows that:

$$E(T) = (0.9)^{10} + 11(1 - (0.9)^{10}) = 7.513$$

6 Ex 46 p.180 (5 points)

The first step is to correctly setup the problem in terms of the given probabilities. Let the following be the events that we are interested in:

C= Event that someone is Convicted by a single juror.
A= Event that someone is Acquitted by a single juror.
G= Event that someone is Guilty.
I= Event that someone is Innocent.

Moreover we know that the jurors make their decisions independently of each other and we assume that the decision process of each juror is the same, i.e. for each juror the following holds:

$$P(C \mid I) = 0.1$$
$$P(A \mid I) = 0.9$$
$$P(A \mid G) = 0.2$$
$$P(C \mid G) = 0.8$$

Finally we know that a defendant will be convicted if the number of 'guilty' votes from the 12 member jury will be larger than or equal to 9. If we consider Z as the total number of votes for conviction for someone on trial, then Z is nothing but a Binomial r.v. and the following is true:

Conviction when Guilty : $P(Z \geq 9 \mid G) = \sum_{i=9}^{12} \binom{12}{i} 0.8^i 0.2^{12-i} = 0.79457$
Conviction when Innocent : $P(Z \geq 9 \mid I) = \sum_{i=9}^{12} \binom{12}{i} 0.1^i 0.9^{12-i} = 1.65835e - 07$
Acquittal when Innocent : $P(Z < 9 \mid I) = 1 - 1.65835e - 07 = 0.9999998$

So then for the probability of a correct decision we just need $P(G) * P(Z \geq 9 \mid G) + P(I) * (1 - P(Z \geq 9 \mid I)) = 0.65 * 0.7945 + 0.35 * 0.9999998 = 0.8665$ (Sorry to those who did this correctly and got it marked wrong, I made a mistake in grading. If this applies to you please resubmit your HW3 with HW4 and I will make the necessary adjustment to your grade).

For the probability of a conviction we just need $P(G) * P(Z \geq 9 \mid G) + P(I) * P(Z \geq 9 \mid I) = P(G) * P(Z \geq \mid G) + P(I) * P(Z \geq 9 \mid I) = 0.65 * 0.7945 + 0.35 * 1.65835e - 07 = 0.5164$

7 Ex 49 p.181 (3 points)

It will be a Binomial r.v. if and only if chip defects are independent and each one has the same probability of occurring. In this case the r.v. X: *number of defective chips* is actually Binomial with parameters $n = 100$ and $p = 0.1$ (It is doubtful in this case that we actually have independence. Why?).

8 Ex 74 p.183 (4 points)

(a) $\frac{\binom{94}{10}}{\binom{100}{10}} = 0.5223$

(b) $\frac{\binom{94}{10}}{\binom{100}{10}} + \frac{\binom{94}{9}\binom{6}{1}}{\binom{100}{10}} + \frac{\binom{94}{8}\binom{6}{2}}{\binom{100}{10}} = 0.01255$

9 Ex 3 p.180/p.185 (5th edition) (4 points)

$P(X \geq a) = 1 - P(X < a) = 1 - (P(X \leq a) - P(X = a)) = 1 - \lim_{h \downarrow 0} F(a - h)$

10 Ex 4 p.185 (5th edition) (4 points)

A possible counterexample is given by a constant random variable $X \equiv b$ with the sequence:

$$b_n = b + \frac{1}{n}$$

For this random variable and this sequence,

$$P(X < b) = 0 \neq 1 = \lim_{b_n \to b} P(X < b_n)$$

Ex 4 p.180 (6th edition) (4 points)

Let $Y = e^X$. Then $F_y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \log(y)) = F_x(\log(y))$, for $y > 0$.

11 Ex 20 p.186 (5th edition) (5 points)

$$E(X!) = \sum_{x=0}^{\infty} x! \frac{e^{-\lambda}\lambda}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \lambda^x = \begin{cases} \frac{e^{-\lambda}}{1-\lambda} & 0 < \lambda < 1 \\ \infty & \lambda \geq 1 \end{cases}$$

Ex 20 p.181 (6th edition) (5 points)

B. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a Poisson random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S > 0$,

$$P(X = 1) = P(S = 1|S > 0) = \frac{P(S = 1)}{P(S > 0)} = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

12 Another exercise (5 points)

For the martingale strategy X has the following probability function:

$$\begin{cases} 1 & \text{with probability} = 1 - (\frac{20}{38})^4 \\ -15 & \text{with probability} = (\frac{20}{38})^4 \end{cases}$$

It follows that:
1) $P(X > 0) = 1 - (\frac{20}{38})^4 = 0.9232664$
2) This could be thought of as a "winning" strategy in that the probability that you win \$1 is very high. However, because if you lose, you lose \$15, some may not want to play this "winning" strategy because as shown in part 3) below, on average you will lose about \$0.23.
3) $E(X) = 0.9232664 - 15 * 0.0767336 = -0.2277376$