Sta 104 Homework 7 solutions

November 5, 2001

All problems are worth 4 points unless noted.

Problem 5 p.379/372ff

The joint density of the point (X, Y) at which the accident occurs is

$$f(x,y) = \begin{cases} \frac{1}{9}, & -3/2 < x, y < 3/2 \\ 0, & \text{otherwise} \end{cases}$$

$$= f(x)f(y)$$

where

$$f(a) = \begin{cases} \frac{1}{3}, & -3/2 < a < 3/2 \\ 0, & \text{otherwise} \end{cases}$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$. Therefore,

$$E(|X| + |Y|) = \frac{2}{3}\int_{-3/2}^{3/2} |x| dx = \frac{4}{3}\int_{0}^{3/2} x dx = 3/2$$

Problem 9 p.379/372ff

Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$$E[X_j] = P(\text{ball } i \text{ is not in urn } j,\, i \geq j) = \prod_{i=j}^{n}(1 - 1/i)$$

Hence,

(a) $E[\text{number of empty urns}] = \sum_{j=1}^{n} \prod_{i=j}^{n}(1 - 1/i) = \frac{n-1}{2}$

(b) $P(\text{none are empty}) = P(\text{ball } j \text{ is in urn } j, \text{ for all } j) = \prod_{j=1}^{n} 1/j = \frac{1}{n!}$

Problem 12 p.379/372*ff*

Let $m = n$ (for those who use the 5^{th} edition).

(a) Let X_i equal 1 if the person in position i is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \frac{1}{2}\frac{n}{2n-1}, & \text{if } i = 1, 2n \\ \frac{1}{2}\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore, $E\left[\sum_{i=1}^{2n} X_i\right] = \sum_{i=1}^{2n} E[X_i] = \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) = \frac{3n^2-n}{4n-2} = \left(\frac{mn(2n+m-1)}{(m+n)(m+n-1)} \text{ for } 5_{th} \text{ edition}\right)$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result:

$$n\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2} = \left(\frac{2mn(n-1) + nm(m-1)}{(m+n)(m+n-1)} \text{ for } 5_{th} \text{ edition}\right)$$

where the right side equality assumes that $n > 1$.

Problem 14 p.379/372*ff*

The number of stages is a negative binomial random variable with parameters m and $1-p$. Hence, its expected value is $m/(1-p)$.

Problem 19 p.379/372*ff*

(a) E[time of first type 1 catch] $= \frac{1-p_1}{p_1}$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1, & \text{if a type } j \text{ is caught before a type 1} \\ 0, & \text{otherwise} \end{cases}$$

Then $E\left[\sum_{j\neq 1} X_j\right] = \sum_{j\neq 1} E[X_j] = \sum_{j\neq 1} P(\text{type } j \text{ before type 1}) = \sum_{j\neq 1} P_j/(P_j + P_1)$.

Where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give

$P(\text{type } j \text{ before type 1}) = P(j \mid j \text{ or } i) = \frac{P_j}{P_j + P_1}$

Problem 46/42 p.379/372*ff*

$E[I_1 I_2] = \sum_{i=2}^{12} E[I_1 I_2 \mid \text{bankd rolls } i] P(\text{bank rolls } i)$
$= \sum_i (P(\text{roll is greater that } i)^2 P(\text{bank rolls } i)$
$= E[I_1^2] \geq (E[I_1])^2 = E[I_1]E[I_2]$

Problem 51/47 p.379/372*ff*

$f_{X|Y}(x|y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y dx} = \frac{1}{y}, 0 < x < y$
$E[X^3|Y = y] = \int_0^y x^3 \frac{1}{y} dx = \frac{y^3}{4}$

Problem 42 sixth edition p.379/372*ff*

(a) Let

$$X_i = \begin{cases} 1, & \text{if pair } i \text{ consists of a man and a woman} \\ 0, & \text{otherwise} \end{cases}$$

$E[X_i] = P(X_i = 1) = \frac{10}{19}$
$= E[X_i X_j] = P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1)$
$= \frac{10}{19}\frac{9}{17}, i \neq j$

$E\left[\sum_1^{10} X_i\right] = \frac{100}{19} = 5.263$

$Var\left[\sum_1^{10} X_i\right] = 10\frac{10}{19}\left(1 - \frac{10}{19}\right) + 10 \cdot 9\left(\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right) = \frac{900}{19^2}\frac{18}{17} = \frac{16200}{6137} =$ 2.639

(b)

$$X_i = \begin{cases} 1, & \text{if pair } i \text{ consists of a married couple} \\ 0, & \text{otherwise} \end{cases}$$

$E[X_i]n = \frac{1}{19}, E[X_i X_j] = P(X_i = 1)P(X_j = 1|X_i = 1) = \frac{1}{19}\frac{1}{17}, i \neq j$

$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$

$Var\left[\sum_1^{10} X_i\right] = 10\frac{1}{19}\frac{15}{19} + 10 \cdot 9\left(\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right) = \frac{180}{19^2}\frac{18}{17} = \frac{3240}{6137} = 0.528$

Problem 47 sixth edition p.379/372*ff*

(a) It is binomial with parameters $n-1$ and p.

(b) Let $x_{i,j}$ equal 1 if there is an edge between verticies i and j, and let it be 0 otherwise. Then, $D_i = \sum_{k \neq i} X_{i,k}$, and so, for $i \neq j$ $Cov(D_i, D_j) =$
$Cov\left(\sum_{k\neq i} X_{i,k}, \sum_{r\neq j} X_{r,j}\right)$
$= \sum_{k\neq i}\sum_{r\neq j} Cov(X_{i,k}, X_{r,j})$
$= Cov(X_{i,j}, X_{i,j})$
$= Var(X_{i,j})$
$= p(1-p)$

where the third equality uses the fact that except when $k = j$ and $r = i$, $X_{i,k}$ and $X_{r,j}$ are independent and thus have covariance equal to 0. Hence, from part(a) and the preceding we obtain that for $i \neq j$,

$$\rho(D_i, D_j) = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

Exercise 1 p.389/384*ff*

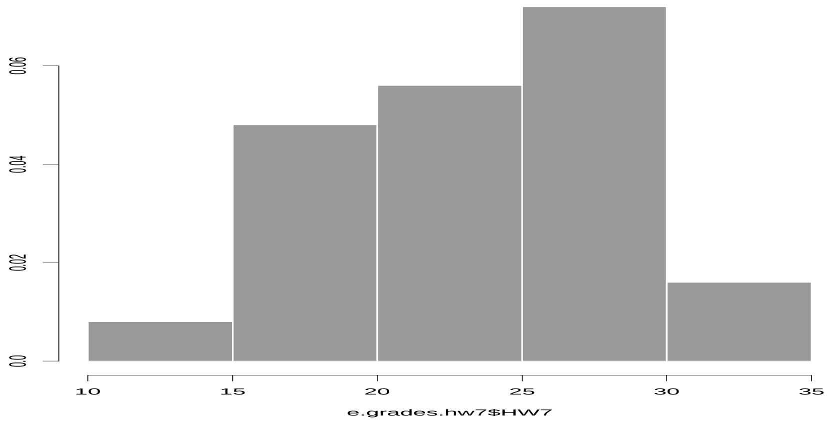
$\frac{d}{da}E[(X-a)^2] = \frac{d}{da}[E(X^2)-2aE(X)+a^2] = -2E(X)+2a$. Setting this equal to zero gives a solution of $a = E(X)$ as either a maximum or a minimum. We check the second derivative to determin which, $\frac{d^2}{da^2}E[(X-a)^2] = \frac{d}{da}[-2E(X)+2a] =$ $2 > 0$, which implies that we have a minimum.

Another Problem

None this week.

Distribution for Homework 7

32 total points possible



	Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max
Raw Score	11.00	20.00	25.00	24.28	29.00	31.00
Percentage	0.344	0.625	0.781	0.759	0.906	0.969