Sta 104 Homework 9 Solutions

November 20, 2001

All problems are worth 4 points unless noted.

Problem 1

The smallest circle in which both points lie is the one having those points both on the circle and diametrically opposed to one another. Say that these points have coordinates (X_1, Y_1) and (X_2, Y_2). Then the area of the circle is:

$$\pi r^2 = \pi \frac{D^2}{4} = \frac{\pi}{4}\left[(X_1 - X_2)^2 + (Y_1 - Y_2)^2\right]$$

where $X_1,\ X_2,\ Y_1,\ Y_2$ are independent uniform random variables on $(0, 1)$. Therefore the expected value of the area of the circle is:

$$\begin{aligned} E(A) &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{\pi}{4}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right] dx_1 dx_2 dx_3 dx_4 \\ &= \int_0^1 \int_0^1 \frac{\pi}{2}(x_1 - x_2)^2 dx_1 dx_2, \text{ by symmetry} \\ &= \frac{\pi}{12} \end{aligned}$$

Problem 2 (10 points)

(a) Joint density functions integrate to 1, since we have a uniform distribution, we know that $f(X,Y) = \frac{1}{\text{area of triangle}} = \frac{1}{1} = 1$. Or straight forward integration gives:

$$\int_0^1 \int_0^{2-2x} f(x,y)\, dy\, dx = 1 \Rightarrow f(x,y) = 1$$

(b) $f(X|Y) = \frac{f(X,Y)}{f(Y)}$. This gives us the following:

$$f(X \mid Y) = \frac{2}{2-Y} \text{ for } 0 < X < 1 - \frac{Y}{2}$$

For $Y = 1$ the result is:

$$f(X \mid Y = 1) = 2 \text{ for } 0 < X < \frac{1}{2}$$

and the conditional expected value is:

$$E(X \mid Y = 1) = \frac{1}{4}$$

Equivalently the conditional expected value can be calculated as:

$$E(X \mid Y) = \frac{2 - y}{4} \text{ for } 0 < Y < 2$$

and the same result as above is obtained by plugging in $y = 1$

(c)
$$E(XY) = \int_0^2 \int_0^{1-y/2} xy \, dx \, dy = \int_0^1 \int_0^{2-2x} xy \, dy \, dx = \frac{1}{6}$$

(d) The line $X = Y$ intersects the triangle's hypotenuse ($Y = 2 - 2X$) at the point $(\frac{2}{3}, \frac{2}{3})$. Then $P(X < Y)$ is the area of the triangle with corners $(0, 0)$, $(0, 2)$ and $(\frac{2}{3}, \frac{2}{3})$. Therefore its value is:

$$P(X < Y) = 2 \cdot \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$$

(e) The line $zX = Y$ intersects the triangle's hypotenuse ($Y = 2 - 2X$) at the point $(\frac{2}{2+z}, \frac{2z}{2+z})$. $P(Z \leq z)$ is the area of the triangle with corners $(0, 0)$, $(1, 0)$ and $(\frac{2}{2+z}, \frac{2z}{2+z})$,which can be calculated as:

$$P(Z \leq z) = \frac{1}{2} \cdot \frac{2z}{2+z} = \frac{z}{2+z} \text{ for } 0 < z < \infty$$

and zero otherwise. The density is the derivative of the above CDF over the domain of Z:

$$f(z) = \frac{2}{(2+z)^2}, \text{ for } z > 0.$$

Problem 3

The probability of observing one of the events $\{1, 2, 3, 4, 5, 6\}$ by rolling a fair die is $\frac{1}{6}$. Assuming them independent, the probability of observing three equal numbers (a *success*) is:

$$6\left(\frac{1}{6}\right)^3 = \frac{1}{36}$$

so that the r.v. *number of rolls until the first success* is Geometric with probability $p = \frac{1}{36}$. Therefore its expected value is $\frac{1}{p} = 36$

Problem 4 (7 points)

a) Binomial with parameters $(n = 25, p = 0.2)$.

b) With n=*number of characters on one page*, since $p = 0.001$ is the probability of an error, the distribution is approximately Poisson with parameter $\lambda = n * 0.001$.

c) Binomial with parameters $(n = 100, p = e^{-1})$.

d) Either Geometric with parameter $p = e^{-1}$ or Negative Binomial with parameters $(\alpha = 1, p = e^{-1})$.

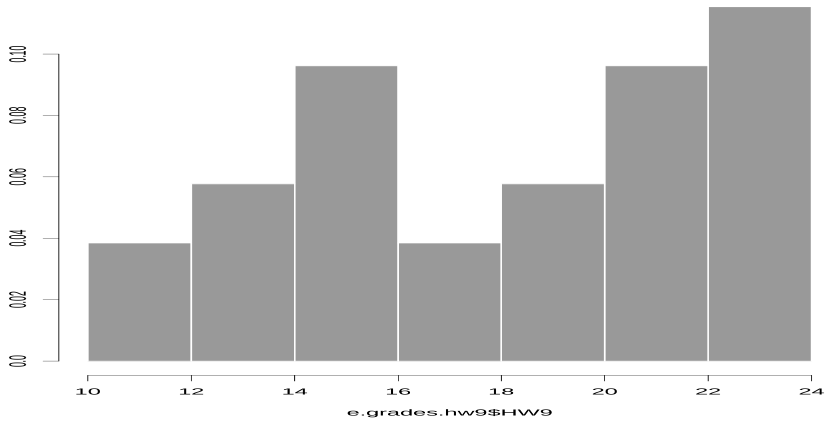
e) The only important characteristics that this density should have are: continuity and positivity $(P(X > 0) = 1)$. A good proposal is the Exponential density with parameter λ= *"average number of buses per hour"*.

f) Poisson with parameter $\frac{\lambda}{6}$, where λ is the average number of buses per hour

g) It can be either a continuous Uniform r.v. with support [0,60) or a discrete Uniform r.v. with support $\{0,...,59\}$.

Distribution for Homework 8

25 total points possible



	Min	**1^{st} Qu.**	**Median**	**Mean**	**3^{rd} Qu.**	**Max**
Raw Score	10	16	19	18.58	22	24
Percentage	0.40	0.64	0.76	0.7431	0.88	0.96