Inference for Stochastic Processes
3. Diffusion Processes

By **Robert L. Wolpert**

Duke University ISDS, USA

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Brownian Motion

- $\omega_0 \equiv 0$, $\omega_1 \sim \mathsf{No}(0,1)$
- $\omega_{1/2} \sim \mathsf{No}\left(\frac{\omega_0+\omega_1}{2}, 1/4\right)$
- $\omega_{1/4} \sim \mathsf{No}\left(\frac{\omega_0+\omega_{1/2}}{2}, 1/16\right) \quad \omega_{3/4} \sim \mathsf{No}\left(\frac{\omega_{1/2}+\omega_1}{2}, 1/16\right)$
- $\omega_{j/2^n} \sim \mathsf{No}\left(\frac{\omega_{\frac{j-1}{2^n}}+\omega_{\frac{j+1}{2^n}}}{2}, \frac{1}{2^{2n}}\right), \qquad j = 1, 3, ..., 2^n - 1$
- $\omega_t \sim \mathsf{No}(0,t), \qquad \mathsf{Cov}(\omega_s, \omega_t) = s \wedge t$

Brownian Motion

- $\Omega = \mathcal{C}\big([0,1]\big)$, Continuous Functions w/sup norm
- $M_t = \omega_t$ and $Q_t = {\omega_t}^2 - t$ both Martingales

Properties:

- ω_t is a Gaussian Markov process
- Independent Increments $[\omega_t - \omega_s]$
- $|\omega_t - \omega_s| \approx c|t-s|^{1/2} \log\log \frac{1}{|t-s|} \Longrightarrow$
  Continuous but nondifferentiable at every point
- $\sum_{i/n \le t} |\omega_{(i+1)/n} - \omega_{i/n}|^2 \to t$ (Quadratic Variation)
- $M_t^\phi \equiv \phi(\omega_t) - \phi(0) - \int_0^t \frac{1}{2}\phi''(\omega_s)\,ds$ is a martingale $\forall \phi \in \mathcal{C}_b^2(\mathbb{R})$
  (Consider $\phi(x) = x$ and $\phi(x) = x^2$)

Brownian Motion Scaling and Reflection: If $c \in \mathbb{R}$, $h > 0$, and ω is a Wiener process, then so are each of:

- $X_1(t) = c\omega_{t/c}$
- $X_2(t) = t\omega_{1/t}$
- $X_3(t) = \omega_{t+h} - \omega_h$
- $X_4(t) = (t+1)\omega_{1/(1+t)} - \omega_1$

Note that $X_4(t)$ is well-defined for all $0 \le t < \infty$ if ω_s is defined for $0 \le s \le 1$; thus, henceforth, we take $\Omega = \mathcal{C}(\mathbb{R}_+)$ (with Borel sets $\mathcal{B}$ for the topology of uniform convergence on compact subsets) and take ω_t to be well-defined for all $t \ge 0$.

Stochastic Integration 1: Wiener Integrals
(Deterministic Integrand)

- $f = \sum c_n 1_{(a_n,b_n]}, a_n, b_n, c_n \in \mathbb{R} \Longrightarrow$ $\int_0^t f(s)\,d\omega_s \equiv \sum c_n[\omega_{b_n\wedge t} - \omega_{a_n\wedge t}]$
- $\mathsf{E}\big[(\int_0^t f(s)\,d\omega_s - \int_0^t g(s)\,d\omega_s)^2\big] = \int_0^t \big(f(s)-g(s)\big)^2\,ds$
- By L^2 continuity, $\int_0^t f(s)\,d\omega_s$ for **all** $f \in L^2_{\text{loc}}$
- $M_t = \int_0^t f(s)\,d\omega_s$ is an L^2 martingale

Stochastic Integration 2: Itô Integrals
(Previsible Integrand)

- $f = \sum c_n 1_{(a_n,b_n]}, a_n, b_n \in \mathbb{R}, \mathbf{c_n} \in \mathbf{L^2}(\Omega, \mathcal{F}_{\mathbf{a_n}}, \mathbf{P}) \Longrightarrow$ $\int_0^t f(s)\,d\omega_s \equiv \sum c_n[\omega_{b_n\wedge t} - \omega_{a_n\wedge t}]$
- $\mathsf{E}\big[(\int_0^t f(s)\,d\omega_s - \int_0^t g(s)\,d\omega_s)^2\big] = \int_0^t \mathsf{E}\big[(f(s)-g(s)\big]^2\,ds$
- By L^2 continuity, $\int f(s)\,d\omega_s$ for L^2 closure of left-continuous step functions $f \in L^2(\mathbb{R}_+ \times \Omega)$— "*previsible processes*".
- **Previsible** processes are closure in $L^2(\mathbb{R}_+ \times \Omega)$ of **left continuous** adapted processes $f(s)$.
- $M_t = \int_0^t f(s)\,d\omega_s$ is (still) an L^2 martingale

Diffusion Processes
The following are equivalent:

- Path-continuous strong Markov process
- Continuous Markov processes with $\mathsf{E}[X_{t+\epsilon} - X_t] = \epsilon a_t(X_t) + o(\epsilon)$, $\mathsf{V}[X_{t+\epsilon} - X_t] = \epsilon b_t^2(X_t) + o(\epsilon)$.
- Process for which

$$M_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t L_s(X_s)\phi(X_s)\,ds$$

is a martingale for all $\phi \in \mathcal{C}_b^2(\mathbb{R})$, where

$$L_s(x)\phi(x) \equiv a_s(x)\phi'(x) + \frac{b_s^2(x)}{2}\phi''(x)$$

- Solution to SDE

$$X_t = X_0 + \int_0^t a_s(X_s)\,ds + \int_0^t b_s(X_s)\,d\omega_s$$

Stochastic Differential Equations (SDEs)

- Classical:

$$X_t = X_0 + \int_0^t a_s(X_s)\,ds + \int_0^t b_s(X_s)\,d\omega_s$$

- Modern:

$$M_t^\phi \equiv \phi(X_t) - \phi(X_0) - \int_0^t L_s\phi(X_s)\,ds$$

is a martingale $\forall \phi \in \mathcal{C}_b^2(\mathbb{R})$, where

$$L_s\phi(x) \equiv a_s(x)\phi'(x) + \frac{1}{2}{b_s}^2(x)\phi''(x)$$

Quadratic Variation

$$\begin{aligned}
[X]_t &= \lim_{\epsilon\to 0} \sum_{j=0}^{\lfloor t/\epsilon\rfloor} (X_{(j+1)\epsilon} - X_{j\epsilon})^2 \\
&= \lim_{\epsilon\to 0} \sum_{j=0}^{\lfloor t/\epsilon\rfloor} \left(a_{j\epsilon}(X_{j\epsilon})\epsilon + b_{j\epsilon}(X_{j\epsilon}) \cdot [\omega_{(j+1)\epsilon} - \omega_{j\epsilon}]\right)^2 \\
&= \lim_{\epsilon\to 0} \sum_{j=0}^{\lfloor t/\epsilon\rfloor} b_{j\epsilon}^2(X_{j\epsilon})\epsilon \\
&= \int_0^t b_s^2(X_s)\, ds
\end{aligned}$$

Properties of Diffusions (SDEs)

- Every continuous strong Markov process is of this form
- Both definitions extend to **Multidimensional** SDEs, *i.e.*, $\mathbb{R}^n$-valued diffusions with $a_s(x) \in \mathbb{R}^n$ and $b_s(x)$ an $n \times n$ matrix
- Both definitions extend to **Martingale** SDEs, where the Wiener process ω_t is replaced by an arbitrary L^2 martingale M_t
- **Manifold**-valued diffusions (on sphere, torus, *etc.*) are also possible (easiest with Modern formulation).
- **Notation**: Common short-hand notation is

$$dX_t = a_t\, dt + b_t\, d\omega_t;$$

meaning is the same as the stochastic **integral** equations

$$X_t = X_0 + \int_0^t a_s(X_s)\, ds + \int_0^t b_s(X_s)\, d\omega_s$$

Examples

- Brownian Motion with Drift:

$$X_t = X_0 + a\,t + b\,\omega_t$$

$a_s \equiv a$ and $b_s \equiv b$ constant: $X_t \sim \mathsf{No}\big(x_0 + at, b(s \wedge t)\big)$

- Brownian Bridge:

$$X_t = X_0 - \int_0^t \frac{X_s}{1-s}\, ds + \omega_t$$

$a_s = \frac{-x}{1-s}$, $b_s \equiv 1$; $\mathsf{P}[X_1 = 0] = 1$: $X_t \sim \mathsf{No}\big(0, s \wedge t - st)\big)$

- Orstein-Uhlenbeck Velocity:

$$X_t = X_0 - \int_0^t \beta\, X_s\, ds + \omega_t$$

$a_s = -\beta x$, $b_s \equiv 1$; stationary Gaussian Markov proc. w/ exponential return to 0: $X_t \sim \mathsf{No}\big(0, \frac{1}{2\beta} e^{-\beta|s-t|}\big)$

- Geometric Brownian Motion:

$$X_t = X_0 + (a + \frac{b^2}{2}) \int_0^t X_s\, ds + \int_0^t X_s\, d\omega_s$$

$a_s = (a + \frac{b^2}{2})x$, $b_s = x$; can be represented $X_t = \exp(Y_t)$ for Brownian Motion with Drift $Y_t = Y_0 + at + b\omega_t$.

- Gaussian if (and only if) $a_s(x) = a_s \cdot x$, $b_s(x) = b_s$
- SDE Notation:
  - In General: $dX_t = a_t(X_t)\, dt + b_t(X_t)\, d\omega_t$
  - Brownian Motion w/Drift: $dX_t = a\, dt + b\, d\omega_t$
  - Brownian Bridge: $dX_t = -\beta X_t/(1-t)\, dt + d\omega_t$
  - O-U Velocity: $dX_t = -\beta X_t\, dt + \, d\omega_t$
  - Geometric BM: $dX_t = (a + \frac{b^2}{2}) X_t\, dt + X_t\, d\omega_t$

Inference for Brownian Motion w/Drift

- Likelihood wrt Lebesgue: For $n \in \mathbb{N}$,

$$\begin{aligned} L_n(a,b) &= \prod_{i=0}^{n-1} \left[\frac{1}{\sqrt{2\pi b^2/n}} \exp\left\{ \frac{(X_{(i+1)/n} - X_{i/n} - a/n)^2}{2b^2/n} \right\} \right] \\ -\log L_n(a,b) &= \frac{n}{2} \log \frac{2\pi b^2}{n} + \frac{n}{2b^2} \sum \left(X_{(i+1)/n} - X_{i/n} - a/n \right)^2 \\ &= \frac{n}{2} \log \frac{2\pi b^2}{n} + \frac{n}{2b^2} \sum \left(\Delta_i - a/n \right)^2 \\ &= \frac{n}{2} \log \frac{2\pi b^2}{n} + \frac{n}{2b^2} \sum \Delta_i^2 - \frac{a}{b^2} X_1 + \frac{a^2}{2b^2} \end{aligned}$$

- Relative Likelihood: For $n \in \mathbb{N}$, if $H_0 : X_t \sim \mathsf{BM}(a_0, b_0)$,

$$\begin{aligned} -\log L_n(a_j,b_j) &= \frac{n}{2} \log \frac{2\pi b_j^2}{n} + \frac{n}{2b_j^2} \sum \Delta_i^2 - \frac{a_j}{b_j^2} X_1 + \frac{a_j^2}{2b_j^2} \\ \log \frac{L_n(a_1,b_1)}{L_n(a_0,b_0)} &= \frac{n}{2} \log \frac{b_0^2}{b_1^2} + \frac{n}{2} \left(\frac{1}{b_0^2} - \frac{1}{b_1^2} \right) \sum \Delta_i^2 \\ &\quad + \left(\frac{a_1}{b_1^2} - \frac{a_0}{b_0^2} \right) X_1 + \frac{1}{2} \left(\frac{a_0^2}{b_0^2} - \frac{a_1^2}{b_1^2} \right) \\ &\approx \frac{n}{2} \left(1 - \frac{b_0^2}{b_1^2} + \log \frac{b_0^2}{b_1^2} \right) + \left(\frac{a_1}{b_1^2} - \frac{a_0}{b_0^2} \right) X_1 \\ &\quad - \frac{1}{2} \left(\frac{a_1^2}{b_1^2} - \frac{a_0^2}{b_0^2} \right) \\ &\to \begin{cases} -\infty & \text{if } b_0 \neq b_1 \\ \frac{a_1 - a_0}{b_0^2} X_1 - \frac{a_1^2 - a_0^2}{2\, b_0^2} & \text{if } b_0 = b_1 \end{cases} \end{aligned}$$

Conclusions about Brownian Motion w/Drift

- $b_1 \neq b_0 \Longrightarrow \log \frac{L(a_1,b_1)}{L(a_0,b_0)} \to -\infty$ as $n \to \infty$, so we can estimate b perfectly;
- $b_1 = b_0 \Longrightarrow \log \frac{L(a_1,b_1)}{L(a_0,b_0)} = \frac{a_1 - a_0}{b_0^2} \left[X_1 - \frac{a_0 + a_1}{2} \right]$, so we cannot estimate a perfectly. Rewriting with $a_0 = \alpha$, $b_0 = b_1 = \beta$, $a_1 = a$,

$$\begin{aligned} \log \frac{L(a,\beta)}{L(\alpha,\beta)} &= \frac{a - \alpha}{\beta^2} \left[X_1 - \frac{a + \alpha}{2} \right] \\ &= -\frac{(a - X_1)^2}{2\beta^2} + \frac{(\alpha - X_1)^2}{2\beta^2} \end{aligned}$$

so $a\big|_{\{X_t\}} \sim \mathsf{No}(X_1, \beta^2)$ *a posteriori*, for a reference analysis with uniform prior.

Extensions From the quadratic variation $[X]_t$ we can always identify $b_t^2(X_t)$ exactly for any diffusion

$$X_t = X_0^t + \int_0^t a_s(X_s)\, ds + \int_0^t b_s(X_s)\, d\omega_s$$