STA 290 STATISTICAL LABORATORY **HOMEWORK 2**
Fall Semester 2002

Solutions should be written up using Latex. Include graphs only where required or (sparingly) where they will help with your discussion of solutions. You may include graphs in text or as an appendix.

Exercise (1)

Assume the result from standard advanced calculus for the complete Beta integral

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(a) If X has a Beta distribution with parameters α and β, find the $E(X^m(1-X))^n$

(b) Use this to find the mean and variance of X.

Exercise (2)

Suppose that Y has a binomial distribution for fixed n and unknown parameter θ, where the prior distribution for θ is Beta(α, β).

(a) Find the marginal distribution of Y, $p(y)$, for $y = 0, \ldots, n$ unconditional on θ, i.e.

$$p(y = k) = \int_0^1 P(y = k|\theta)p(\theta)d\theta)$$

where $P(y = k|\theta)$ is the binomial mass function of y given θ and $p(\theta)$ is the prior density of θ. This is the prior predictive distribution. This discrete distribution is known as the beta-binomial distribution.

(b) Find the unconditional variance of Y. (how does this compare to the Binomial variance?)

(c) Show that if the beta-binomial distribution probability is constant in y, then the prior distribution has to have $\alpha = \beta = 1$.

(d) Show that the posterior mean of θ is a weighted average of the prior mean $(\alpha/(\alpha+\beta)$ and the sample proportion y/n.

(e) Show that if prior distribution on θ is Uniform, then the posterior variance is always less than the prior variance of θ.

(f) Give an example of another Beta(α, β) prior distribution and data y and n in which the posterior variance of θ is higher than the prior variance. (uncertainty has increased given the data)

Exercise (3)

Suppose that the proportion of Californians who support the death penalty has a beta distribution with mean 0.6 and standard deviation 0.3

(a) Determine the parameters α and β of the prior distribution. Graph the prior density function.

(b) A random sample of 1000 Californians is taken and 65% support the death penalty. Find the posterior distribution for the proportion that support the death penalty, as well as the mean and posterior standard deviation. Add the posterior density to the graph of the prior density.

Exercise (4)

(a) Show that the Binomial distribution $\text{Bin}(n, \theta)$ is an exponential family with natural parameter $\text{logit}(\theta) \equiv \log(\theta/(1-\theta)$

(b) For the binomial likelihood, $y \sim \text{Bin}(n, \theta)$, show that $p(\theta) \propto \theta^{-1}(1-\theta)^{-1}$ is the uniform distribution for the natural parameter of the exponential family.

(c) Find the posterior distribution for θ using the above prior distribution.

(d) Show that if $y = 0$ or if $y = n$, the resulting posterior distribution is improper.

Exercise (5)

Let θ denote the probability of a birth being female, and suppose you have assigned a Beta(100,100) prior distribution to θ

(a) Show that this means that you are more than 95% sure that θ is between 0.4 and 0.6, although you are ambivalent as to whether it is greater or less than 0.5

(b) Now you observe a random sample of 1000 births, 511 are boys. What is your posterior probability that θ is greater than 0.5?
