INTRODUCING NORMAL MODELS

Readings: GCSR Sec 2.6, 2.8, Chapter 3.

IID observations $Y = (Y_1, Y_2, \ldots Y_n)$

$$Y_i \sim N(\mu, \sigma^2)$$

unknown parameters μ and σ^2.

From a Bayesian perspective, it is easier to work with the *precision*, ϕ, where $\phi = 1/\sigma^2$.

Likelihood

$$\begin{aligned} L(\mu, \phi | Y) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \phi^{1/2} \exp\{-\frac{1}{2}\phi(Y_i - \mu)^2\} \\ &\propto \phi^{n/2} \exp\{-\frac{1}{2}\phi \sum_i (Y_i - \mu)^2\} \end{aligned}$$

Likelihood

$$
\begin{aligned}
L(\mu, \phi | Y) &\propto \phi^{n/2} \exp\{-\frac{1}{2}\phi \sum_i (Y_i - \mu)^2\} \\
&\propto \phi^{n/2} \exp\{-\frac{1}{2}\phi \sum_i \left[(Y_i - \bar{Y}) - (\mu - \bar{Y})^2\right]\} \\
&\propto \phi^{n/2} \exp\{-\frac{1}{2}\phi \left[\sum_i (Y_i - \bar{Y})^2 + n(\mu - \bar{Y})^2\right]\} \\
&\propto \phi^{n/2} \exp\{-\frac{1}{2}\phi s^2 (n-1)\} \exp\{-\frac{1}{2}\phi n(\mu - \bar{Y})^2\}
\end{aligned}
$$

where $s^2 = \sum_i (Y_i - \bar{Y})^2/(n-1)$ is the usual sample variance.

Prior Distributions

Conjugate prior distribution for (ν, ϕ) is Normal-Gamma.

$$\begin{aligned} \mu | \phi &\sim N(\mu_0, 1/(n_0 \phi)) \\ \phi &\sim \text{Gamma}(\nu_0/2, (\nu_0 \phi_0)/2) \end{aligned}$$

$$p(\phi) \propto \phi^{\nu_0/2-1} \exp\{-\phi \nu_0 \phi_0 / 2\}$$

Non-informative prior distribution (improper)

$$p(\mu, \phi) = 1/\phi$$

assuming prior independence of location and scale parameters, μ is uniform on the real line, $\log(\phi)$ is uniform on real line based on Jeffreys' invariance principle (GCSR Sec 2.8).

Posteriors under the Non-Informative Prior

$$
\begin{aligned}
p(\mu, \phi | Y) &\propto L(\mu, \phi) p(\mu, \phi) \\
&= \phi^{n/2-1} \exp\{-\frac{1}{2}\phi s^2 (n-1)\} \exp\{-\frac{1}{2}\phi n (\mu - \bar{Y})^2\} \\
&= \left\{ \phi^{\frac{n-1}{2}-1} e^{\{-\frac{1}{2}\phi s^2 (n-1)\}} \right\} \left\{ \phi^{1/2} e^{-\frac{1}{2}\phi n (\mu - \bar{Y})^2} \right\} \\
&\propto \text{ Gamma}\left(\frac{n-1}{2}, (n-1)\frac{s^2}{2} \right) N\left(\bar{Y}, \frac{1}{\phi n} \right) \\
&= p(\phi | Y) p(\mu | \phi, Y)
\end{aligned}
$$

Marginal Distribution for $\mu|Y$

Obtain the marginal distribution for μ by integrating out ϕ from the joint posterior distribution, and recognize the kernel of the distribution!

$$
\begin{aligned}
p(\mu|Y) &\propto \int p(\mu, \phi|Y) d\phi \\
&= \int \phi^{n/2-1} \exp\{-\frac{1}{2}\phi s^2(n-1) + n(\mu - \bar{Y})^2\}
\end{aligned}
$$

This has the form of a Gamma integral with $\alpha = n - 1$ and β equal to the mess multiplying ϕ,

$$
\begin{aligned}
p(\mu|Y) &\propto (s^2(n-1) + n(\mu - \bar{Y})^2)^{(n-1+1)/2} \\
&\propto \left(1 + \frac{1}{n-1}\frac{(\mu - \bar{Y})^2}{s^2/n}\right)^{(n-1+1)/2}
\end{aligned}
$$

Student-$t_{n-1}(\bar{Y}, s^2/n)$ location $\bar{Y}$, df = n-1, scale s^2/n)