STA104: solutions to Homework 2

September 16, 2003

Pr 7

a) Each person can fall into one of $2 \times 3 = 6$ categories, so the number of outcomes in the sample space is $6^{15} = 470184984576$

b) The number of ways to have 15 white-collar workers is 3^{15}, so the number with at least one blue-color worker is $6^{15} - 3^{15} = 470170635669$.

c) $4^{15} = 1073741824$

Pr 10

This is Venn Diagram question— it's helpful to draw the picture (or fill in the 2×2 table). 30% wear one or the other, and 10% wear both.

Pr 17

There are 8 rows and 8 columns, for a total of 64 squares. Altogether there are $\binom{64}{8}$ ways to choose eight squares for the rooks.

For the rooks to be safe from each other there must be one rook on each row. The first can be in any of the eight columns; given that choice, the second has seven possibilities, the next six, the next five, and so forth. This leaves us with 8! ways to place the rooks so that none can capture another. Therefore the probability is:

$$P(\text{Rooks safe}) = \frac{8!}{\binom{64}{8}} = \frac{40320}{4426165368} = \frac{560}{61474519} \approx 9.109465 \cdot 10^{-06}.$$

Another approach is to place the eight rooks one at a time. When placing the $k+1$'st rook we must avoid the k rows of its predecessors and the k files of its predecessors, leaving $(8-k)^2$ acceptable locations out of the $64-k$ available ones; thus, multiplying terms for $k = 0..7$,

$$\begin{aligned} P(\text{Rooks safe}) &= \frac{(8-0)^2}{64}\frac{(8-1)^2}{63}\frac{(8-2)^2}{62}\frac{(8-3)^2}{61}\frac{(8-4)^2}{60}\frac{(8-5)^2}{59}\frac{(8-6)^2}{58}\frac{(8-7)^2}{57} \\ &= \frac{8!^2}{8!\binom{64}{8}} \approx 9.109465 \cdot 10^{-06}. \end{aligned}$$

Pr 19

The probability of both red or both black is $(2/6)^2 = 1/9$ and both white or both yellow is $(1/6)^2 = 1/36$. Thus the probability of both the same color is: $1/9 + 1/9 + 1/36 + 1/36 = 5/18 \approx 0.2778$.

Pr 28

The probability of 3 red is $\binom{5}{3}/\binom{19}{3} = 10/969 \approx 0.01032$. The probabilities of choosing 3 green or 3 blue are computed similarly. Thus the probability of 3 of the same color is:

$$\frac{\binom{5}{3}}{\binom{19}{3}} + \frac{\binom{6}{3}}{\binom{19}{3}} + \frac{\binom{8}{3}}{\binom{19}{3}} = \frac{86}{969} \approx 8.875\%.$$

Pr 56

One should choose to go second in this game. Notice that the average of each wheel is 5. This means that whoever chooses first has no obvious basis on which to choose one wheel over another.

If the first player chooses wheel a and the second c, we notice that on wheel a, 9 always wins, 1 always loses, and 5 wins only 1/3 of the time, so a will beat c with probability $(1/3) \times 1 + (1/3) \times 0 + (1/3) \times \frac{1}{3} = 4/9$, and so if the first player chooses a he or she will lose with probability 5/9.

Similarly if the first player chooses b and the second a then the first player will lose with probability 5/9. Finally, if the first player chooses c and the second b, then again the first player will lose with probability 5/9. Evidently the second player has the advantage, no matter which wheel the first player chooses.

Ex 13

Notice that $E = EF \cup EF^c$ Since EF and EF^c are disjoint, we know that $P(EF \cup EF^c) = P(EF) + P(EF^c)$. Therefore, $P(E) = P(EF) + P(EF^c)$. A lot of you wrote things like $P(\text{something}) \cup P(\text{something else})$ or $P(\text{something}) \cap P(\text{something else})$. This does not make sense because probabilities are numbers, and union and intersections work on sets. Similarly, if E and F is a set, then

$E \cdot P(F)$ does not mean anything. Finally, an argument using Venn diagrams can be convincing and is a good way to get the right ideas for the proof. However, this by itself does not qualify as a proof. In general, a proof should consist of a series of statements, each of which is true and follows directly from previous statements. If a statement is incorrect or does not follow from previous statements, then something is missing.

Ex 20

Denote our sample space by S and its countably-many elements by $S = \{s_1, s_2, \cdots\}$. We proceed by contradiction.

Suppose that all the elementary outcomes had the same probability $\epsilon > 0$. Then for any large enough M (any M greater than $1/\epsilon$ will do), the probability of the first M elements would be

$$P\big(\{s_1, ..., s_M\}\big) = \sum_{i=1}^{M} P\big(\{s_i\}\big) = M\epsilon > 1,$$

an impossibility. Thus our supposition that all the elementary outcomes had the same probability $\epsilon > 0$ cannot be true.

Let p_i be any sequence of (strictly) positive numbers that sum to one— for example, consider $p_i = 2^{-i}$. If each elementary outcome s_i has probability p_i, and every event $E \subset S$ probability $P(E) = \sum\{p_i : \ s_i \in E\}$, then we have a probability assignment obeying all three rules with infinitely-many outcomes each with positive probability.

Any other sequence $p_i > 0$ satisfying $\sum p_i = 1$ would work as well— such as $p_i = e^{-1}/i!$, or $6/(\pi i)^2$, or any of the infinitely-many other choices.

Another Problem

Under the given assumptions, each student has probability $364/365$ of *not* sharing my birthday; the probability that every one of these $n = 39$ students do not share my probability is $(364/365)^{39}$, so the chance that at least one *does* share my birthday is

$$1 - \left(\frac{364}{365}\right)^{39} \approx 0.1014707.$$

For a class to have a $50 : 50$ chance of including someone who shares my birthday, the class size n would have to satisfy

$$\begin{aligned}
0.50 &\leq 1-\left(\frac{364}{365}\right)^n, \text{ i.e.} \\
0.50 &\geq \left(\frac{364}{365}\right)^n, \text{ i.e.} \\
\log(1/2) &\geq n\log\left(\frac{364}{365}\right), \text{ i.e.} \\
n &\geq \log(2)/(\log(365)-\log(364)) \\
&\approx 252.65,
\end{aligned}$$

more than half of 365. Does that surprise you?