STA104: solutions to Homework 3

Due Thu, Sept 18, 2003

Pr 7 p.171

Parts (a) and (b) are somewhat confusingly worded; they are asking about the possible values of the *random variable* that is the larger (for (a)) or smaller (for (b)) of the numbers on the two dice— *not* the fixed largest and smallest values (6 and 1) that a die can show. Thus, the answers are:

a) {1,2,3,4,5,6}

b) {1,2,3,4,5,6}

c) {2,3,4,5,6,7,8,9,10,11,12}

d) {-5,-4,-3,-2,-1,0,1,2,3,4,5}

Pr 14 p.174

The problem doesn't give any notation for the number each player gets; I'll call them P_1, P_2, P_3, P_4, and P_5. They are all distinct, and their order is equally likely to be any of the $5! = 120$ possibilities. Now:

- $P[X = 0]$ is the probability $P[P_1 < P_2]$; by symmetry, this must be $1/2 = 60/120$.
- $P[X = 1]$ is the probability that Player 1 has a higher number than Player 2 but a lower number than Player 3, i.e., $P[P_2 < P_1 < P_3]$; since each of the 3! orderings of players 1,2,3 are equally likely, this event has probability $1/6 = 20/120$.
- $P[X = 2]$ is the probability that Player 1 has a higher number than Players 2 or 3 but a lower number than Player 4; this can happen in two different ways (depending on whether or not $P_2 < P_3$), each with probability $1/4! = 1/24$, so $P[X = 2] = P[X_2 < X_3 < X_1 < X_4] + P[X_3 < X_2 < X_1 < X_4] = 2/4! = 1/12$=10/120.

- $P[X = 3]$ is the probability that Player 1 has a higher number than Players 2 or 3 or 4 but a lower number than Player 5, i.e., that player 5 has the largest number and Player 1 has the second-largest number. Each (ordered) pair of players has the same probability of having the largest and second-largest numbers $(3!/5! = 1/20)$, so $P[X = 3] = 1/20 = 6/120$. The last case is easy:

- $P[X = 4]$ is the probability that Player 1 has the largest of the five numbers. This event has probability $P[X = 4] = 1/5 = 24/120$.

As a check we can put all five possibilities into fractions with common denominator 120, and verify

$$\sum_{x=0}^{4} P[X = x] = \frac{60 + 20 + 10 + 6 + 24}{120} = \frac{120}{120} = 1$$

Pr 19

We know that the chance of getting a value within one of the open intervals (without the endpoints) must be zero since the distribution function doesn't change. Since the function jumps, we know that the points with positive probability are at the endpoints: 0, 1, 2, 3, 3.5. Just as in Section 4.2, we can compute those probabilities by the size of the jump– for example, $P(X = 0) = \lim_n P(0 - 1/n < X \leq 0) = \lim_n F(0) - F(0 - 1/n) = 1/2$. Similarly,

$$\begin{array}{lcl} P(X=0) & = & \lim_n F(0) - F(0-1/n) & = & 1/2 \\ P(X=1) & = & \lim_n F(1) - F(1-1/n) & = & 1/10 \\ P(X=2) & = & \lim_n F(2) - F(2-1/n) & = & 1/5 \\ P(X=3) & = & \lim_n F(3) - F(3-1/n) & = & 1/10 \\ P(X=3.5) & = & \lim_n F(3.5) - F(3.5-1/n) & = & 1/10 \end{array}$$

As a check, the total is $\dfrac{5+1+2+1+1}{10} = 1$.

Pr 33

Let X be the "daily demand"; the problem specifies that the distribution of this random variable is Bi(10,1/3).

Let k denote the number of papers he buys.

His net profit, in cents, is his cost minus his sales, i.e. the function

$$15 * min(k, X) - 10 * k$$

Call its expectation $f(k)$; we want to find k to maximize $f(k)$. For example, $f(0) = 0$ (no cost, no sales, no profit), while

$$f(1) = 15 * Pr[X > 0] - 10$$
$$f(2) = 15 * Pr[X = 1] + 30 * Pr[X > 1] - 20$$
$$f(3) = 15 * Pr[X = 1] + 30 * Pr[X = 2] + 45 * Pr[X > 2] - 30$$
$$f(4) = 15 * Pr[X = 1] + 30 * Pr[X = 2] + 45 * Pr[X = 3] + 60 * Pr[X > 3] - 40$$

and so forth.

Look at the successive differences

$f(k) - f(k-1) =$ Additional profit from buying k'th paper $= 15 * Pr[X \geq k] - 10$

We want to buy at least k papers as long as this is positive; the values are easy to compute for the first few k's,

$$f(1) - f(0) = 4.7399 = 15 * (1 - (2/3)^1 0) - 10$$
$$f(2) - f(1) = 3.4393 = 15 * (1 - (2/3)^1 0 - 10 * (1/3) * (2/3)^9) - 10$$
$$f(3) - f(1) = 0.5129 = 15 * (1 - (2/3)^1 0 - 10 * (1/3) * (2/3)^9 - 45 * (1/3)^2 * (2/3)^8) - 10$$
$$f(4) - f(3) = -3.3890$$

We should stop at 3 papers.

Pr 45

$$P(mboxpassing|mboxon, 3testers) = 3 \cdot .8^2 \cdot .2 + .8^3 = .896$$
$$P(mboxpassing|mboxoff, 3testers) = 3 \cdot .4^2 \cdot .6 + .4^3 = .352$$
$$P(mboxpassing|mboxon, 5testers) = 10 \cdot .8^3 \cdot .2^2 + 5 \cdot .8^4 \cdot .2 + .8^5 = .94208$$
$$P(mboxpassing|mboxoff, 5testers) = 10 \cdot .4^3 \cdot .6^2 + 5 \cdot .4^4 \cdot .6 + .4^5 = .31744$$

Alright, we are looking for the highest probability of passing. His chance of an off day is 2/3. This means that for 3 examiners we have a chance of passing of $1/3 \cdot .896 + 2/3 \cdot .352 \approx .533$. If there are 5 examiners we have $1/3 \cdot .942 + 2/3 \cdot /317 \approx .525$.

It's close, but the student should choose 3 examiners.

Pr 49

It will be a Binomial r.v. if and only if chip defects are independent and each one has the same probability of occurring. In this case the r.v. X: *number of defective chips* is actually Binomial with parameters $n = 100$ and $p = 0.1$. One might suspect that it is unlikely that the defects are independent since whatever causes, the defect in one chip is likely to still be there when the next chip is fabricated. On the other hand, if the chips actually sent to you are not in a series (chosen at random from the producer's inventory) then perhaps we do have independence.

Pr 73

(a) $\dfrac{\binom{94}{10}\binom{6}{0}}{\binom{100}{10}} = 0.5223$

(b) $\dfrac{\binom{94}{10}\binom{6}{0}}{\binom{100}{10}} + \dfrac{\binom{94}{9}\binom{6}{1}}{\binom{100}{10}} + \dfrac{\binom{94}{8}\binom{6}{2}}{\binom{100}{10}} = 0.01255$

There was some confusion over whether to do problems 3,5, and 20 from the theoretical exercises or from the self test section. The intention was to ask for theoretical exercises, but the page number was wrong. I accepted either.

Ex 3

$P(X \geq a) = 1 - P(X < a) = 1 - (P(X \leq a) - P(X = a)) = 1 - \lim_{h\downarrow 0} F(a-h)$

Ex 5

We have a new random variable, $Y = \alpha X + \beta$. We are asked to find F_Y given F_X. We know that $F_Y(a) = P(Y \leq a) = P(\alpha X + \beta \leq a) = P(\alpha X \leq a - \beta)$. Now there are two possibilities. If $\alpha > 0$ then $F_Y(a) = P(X \leq (a-\beta)/\alpha) = F_X((a-\beta)/\alpha)$. Otherwise, $F_Y(a) = P(x \geq (a-\beta)/\alpha) = 1 - F_X((a-\beta)/\alpha) + P(X = (a-\beta)/\alpha)$

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Ex 20 B. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a Poisson random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S > 0$,

$$P(X = 1) = P(S = 1|S > 0) = \frac{P(S = 1)}{P(S > 0)} = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

Another exercise

For the martingale strategy X has the following probability function:

$$\begin{cases} 1 & \text{with probability} = 1 - (\frac{20}{38})^4 \\ -15 & \text{with probability} = (\frac{20}{38})^4 \end{cases}$$

It follows that:
1) $P(X > 0) = 1 - (\frac{20}{38})^4 = 0.9232664$
2) This could be thought of as a "winning" strategy in that the probability that you win \$1 is very high. However, because if you lose, you lose \$15, some may not want to play this "winning" strategy because as shown in part 3) below, on average you will lose about \$0.23.
3) If she loses her 28^{th} bet (which is for \$268,435,456), then she is out. Her chance of losing is $(20/38)^{28} \approx 1.6 \times 10^{-8}$. Thus her expected outcome is $\approx 1 - 268435456 * 1.6 \times 10^{-8} = -\$3.20$