Homework 4 solutions

Due Thu, Sept 25, 2003

Problem 2 p.104

Certainly for $i < 7$ we know that this conditional probability must be zero (there aren't any negative numbers on the dice).

- $i = 7$ The first die could be anything from 1 to 6, so p=1/6.
- $i = 8$ The first die could be anything from 2 to 6, so p=1/5.
- $i = 9$ The first die could be anything from 3 to 6, so p=1/4.
- $i = 10$ Possibilities are {5,5}, {6,4}, or {4,6}. p=1/3.
- $i = 11$ There are two ways for this to happen, {5, 6} or {6,5}. Thus p=1/2
- $i = 12$ There is only one way for this to happen, and requires the first dice to be 6, so p=1.

Problem 21 p.106

a) Let's define R_1 and W_1 for Red and White draws from the first urn, and R_2, W_2 for Red, White draws from the second urn. Then:

$$\begin{aligned} P(W_2) &= P(W_1W_2) + P(R_1W_2) \\ &= P(W_1)P(W_2|W_1) + P(R_1)P(W_2|R_1) \\ &= (2/6) * (2/3) + (4/6) * (1/3) \\ &= 2/9 + 2/9 = 4/9 \end{aligned}$$

b)

$$P[W1|W2] = P[W1W2]/P[W2] = (2/9)/(4/9) = 1/2$$

Problem 29 p.107

P(English) = 2/5
P(vowel | english spelling) = 1/2
p(vowel | american spelling) = 2/5
If E is Englishman, A is American, and V is vowel, then:

$$\frac{P(E)\cdot P(V|E)}{P(E)\cdot P(V|E)+P(A)\cdot P(V|A)} = \frac{2/5\cdot 1/2}{2/5\cdot 1/2+3/5\cdot 2/5} = 5/11 \approx 0.4545$$

Problem 47 p.108

We will let S, M, and W stand for strong, moderate and weak recommendations. Also, let J represent the event of being offered the job.

a)

$$\begin{aligned} P(J) &= P(S)P(J|S)+P(M)P(J|M)+P(W)P(J|W) \\ &= 0.7\cdot 0.8+0.2\cdot 0.4+0.1\cdot 0.1 \\ &= 0.65 \end{aligned}$$

b)

$$\begin{aligned} P(S|J) &= P(SJ)/P(J) \\ &= 0.7\cdot 0.8/.65 \approx 0.86 \\ P(M|J) &= P(MJ)/P(J) \\ &= 0.2\cdot 0.4/.65 \approx 0.12 \\ P(W|J) &= P(WJ)/P(J) \\ &= 0.1\cdot 0.1/.65 \approx 0.015 \end{aligned}$$

c)

$$\begin{aligned} P(S|J^c) &= 0.7\cdot 0.2/.35 = 0.4 \\ P(M|J^c) &= 0.2\cdot 0.6/.35 \approx 0.34 \\ P(W|J^c) &= 0.1\cdot 0.9/.35 \approx 0.26 \end{aligned}$$

Problem 50 p.108

These answers are not the only correct ones. Credit was given for having an answer, and backing it up with reasonable explanation.

a) Dependent. If eye color is not the determining factor in whether some one is hired, but is correlated to the determining factor, there would be dependence. For example a white, blue-eyed business woman may prefer to have a white secretary. This excludes all brown-eyed Asians, blacks, etc. who have brown eyes (generally) and significantly increases the chances that the secretary has blue eyes. In a similar manner, a black business woman may prefer a black secretary, both of whom would have the same (or at least similar) eye color.

b) Dependent. Wealth has an effect as a third variable. People that have enough money to buy a car generally also have enough money to have a phone. People who can't afford a phone certainly won't have enough to afford a car either.

c) Dependent, short people generally have less body mass than tall people, so we can assume some correlation between height and weight.

d) Dependent, since the United States is in the Western Hemisphere.

e) Dependence in this case can be seen in seasonal weather patterns. During rainy season, rain today has a strong effect on whether it rains tomorrow. Likewise during dry periods, lack of rain one day strongly effects whether it rains the next.

Problem 60 p.111

To answer this, we need to know what the chances of a correct answer are for each strategy. For strategy **a**, this likelihood is p. For **b**, they will agree on the correct answer with probability p^2. If they disagree, the chance of getting it right is 50%. Thus the probability of arriving at a correct answer is $p^2+p\cdot(1-p)$.

A little algebra shows that $p^2 + p(1-p) = p$. Thus the strategies are equivalent.

Problem 66 p.112

Let Q be the event that the Queen has hemophelia, and for $i = 1, 2, 3, 4$ let P_i be the event that the i'th prince has hemophelia. Then

$$
\begin{aligned}
P[Q|P_1^cP_2^cP_3^c] &= \frac{(1/2)(1/2)^3}{(1/2)(1/2)^3+(1/2)(1)^3} \\
&= \frac{1/16}{1/16+8/16} = \frac{1}{9} \\
P[P_4|P_1^cP_2^cP_3^c] &= (1/2)P[Q|P_1^cP_2^cP_3^c] \\
&= \frac{1}{18}
\end{aligned}
$$

Problem 74 p.113

The chance of an even number is 18/36. There are 6 ways to roll a seven, so the chance of a 7 is 6/36. Rolls other than 7 and evens are irrelevent; just throw them away, and think about the *conditional* probability that the next roll is a seven, *given* that it is either even or a seven:

$$\Pr[S \mid S \cup E] = \frac{6/36}{6/36+18/36} = 6/24 = 1/4$$

So we have two events, S corresponding to getting a seven first ($P(S) = 1/4$), and E corresponding to getting an even number first ($P(E) = 3/4$). What is the probability of getting two S's before 6 E's?

The easier question to answer is the probability of getting 6 E's before 2 S's, i.e., that the number X of sevens before the sixth even number is zero or one. The number of S's before the sixth E has a Negative Binomial distribution, with parameters $r = 6$ and $p = \Pr[E] = 3/4$, so the number X of sevens before the sixth even has probability mass function

$$\Pr[X = x] = \binom{x+6}{x}(3/4)^6(1/4)^x, \qquad x = 0, 1, \ldots$$

and

$$
\begin{aligned}
\Pr[X=0] &= \binom{6}{0}(1/4)^0(3/4)^6 &&= \frac{729}{4096} &&\approx 0.1779785 \\
\Pr[X=1] &= \binom{7}{1}(1/4)^1(3/4)^6 &&= \frac{5103}{16384} &&\approx 0.3114624 \\
\Pr[X<2] & &&= \frac{8019}{16384} &&\approx 0.4894409 \\
\Pr[X\geq 2] &= 1-\Pr[X<2] &&= \frac{8365}{16384} &&\approx 0.5105591,
\end{aligned}
$$

so the probability of getting 2 sevens before 6 even numbers is just over a half, about 51%.

Exercise 2 p.115

IB $A \subset B$,

$$P(A|B) = \frac{P(A)}{P(B)}, \quad P(A|B^c) = 0, \quad P(B|A) = 1, \quad P(B|A^c) = \frac{P(BA^c)}{P(A^c)} = \frac{P(B) - P(A)}{1 - P(A)}$$

Exercise 14 p.117

- To get r successes in n results we need $n - r$ failures. The probability of this happening in any single specific way is $p^r \cdot (1-p)^{n-r}$. The last trial must be a success. This leaves $n-1$ slots for the remaining $r-1$ successes. Thus there are $\binom{n-1}{r-1}$ ways to get the result, for probability $\binom{n-1}{r-1} p^r (1-p)^{n-r}$, for $n = r, r+1, \ldots$.
- Example 4i asks the question: *What is the probability of obtaining a successes before b failures?* This happens in the above situation when $a \leq n < a + b$. We need only add up the probabilities above for n in this range:

$$\sum_{n=a}^{a+b-1} \binom{n-1}{a-1} \cdot p^a \cdot (1-p)^{n-a}$$

is the total probability.
Another way to express this is:

$$P(< m \text{ failures at time of } n^{th} \text{ success}) = \sum_{j=0}^{m-1} \binom{n+j-1}{j} p^n (1-p)^m$$

This solution looks different, but is equivalent to the one in the book.

Another Problem

a) Perhaps surprisingly, the correct answer is **2. Switch to the other unoened door**. This strategy wins with probabily 2/3— any time you pick one of the goats (which happens with probability 2/3), Monte *must* open the door with the other goat, leaving the car behind the remaining closed door. The other 1/3 of the time, when you guessed right and chose the door with the car, Monte chooses one of the goats (the problem doesn't actually specify how he makes that choice) and, whichever he chooses, your decision to switch will win you a goat.

If you always stick to your original choice you will win exactly 1/3 of the time, which is evidently worse.

Try the link 'MH' on the Syllabus webpage (next to the link for this solution set) for an interesting discussion and demonstration applet.

b) Note that the sample space will need more than three points, since Monte has a choice to make (which goat to reveal) whenever you happen to guess the door with the car. One choice is to take all ordered pairs (d_1, d_2) from labels A, B, C for the three doors, where $d_1 = \{\text{door with car behind it}\} \in \{A, B, C\}$ and $d_2 = \{\text{door Monty opens, IF he has a choice}\} \in \{A, B, C\}$, with evidently $d_1 \neq d_2$, so

$$S = \{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}.$$

If Monte chooses at random when he has a choice, and if the car's door is chosen at random, then this is an equally-likely probability space and, whichever door you choose initially, the strategy of **2. Switch to the other unoened door** will win in four of the six cases.