Homework 5 solutions

Due Thu, Oct 2, 2003

Problem 3, page 228

(a) Notice that when $x > \sqrt{2}$, this function is negative. Therefore, it cannot be a probability density function.

(b) Here there is nothing obvious preventing this from being a density function. We'll need to calculate the integral to find C.

$$\int_0^2 C(2x - x^2)dx = C(2 \cdot \frac{2^2}{2} - \frac{2^3}{3}) = C\frac{4}{3}$$

For this to be 1, we need C to be 3/4.

Problem 7, page 229

We need to determine two different constants. We'll need to use the definition of expected value, and the fact that the integral of a density function over the entire space is 1.

$$\begin{array}{llll} 1 & = \int_{-\infty}^{\infty} f(x)\,dx & = \int_0^1 a + bx^2 dx = a \cdot 1 + \frac{b}{3} \cdot 1^3 & = a + \frac{b}{3} \\ \frac{3}{5} & = \int_{-\infty}^{\infty} x\,f(x)\,dx & = \int_0^1 ax + bx^3 dx & = \frac{a}{2} + \frac{b}{4} \end{array}$$

This leaves two linear equations in two unknowns, an easy problem to solve. One way is to eliminate one of the variables: double the second equation and subtract it from the first, giving $b/3 - b/2 = -1/5$; thus $b = 6/5$, and so $a = 3/5$.

Problem 14, page 229

From proposition 2.1,

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f(x)dx \\ &= \int_0^1 x^n dx \\ &= \frac{1}{n+1} x^{n+1}\Big|_{x=0}^{1} = \frac{1}{1+n} \end{aligned}$$

To use the definition, we must calculate the density function for $Y = X^n$.

$$\begin{aligned} F_Y(y) &= \Pr[Y \le y] \\ &= \Pr[X^n \le y] \\ &= \Pr[X \le \sqrt[n]{y}] \qquad \text{for } y \ge 0 \\ &= \int_0^{\sqrt[n]{y}} f(t)dt \end{aligned}$$

We know that $f \sim \mathsf{Un}(0,1)$, so this integral is $\sqrt[n]{y}$ for $0 < y < 1$. Differentiating this, we get a probability density function of $f(y) = (1/n)y^{(1/n)-1}$ on $(0,1)$, $f(y) = 0$ elsewhere. Thus:

$$\begin{aligned} E[Y] &= \int_0^1 y \cdot \frac{1}{n} y^{(1/n)-1} dy \\ &= \frac{1/n}{(1/n)+1} = \frac{1}{1+n}. \end{aligned}$$

Problem 17, page 230

First we need to calculate the probabilities for getting each score on any given shot. Let S be the points scored. Since we are told that the distance is $\mathsf{Un}(0,10)$, we know that:

$$\begin{aligned} \Pr[S = 10] &= \frac{1}{10} \\ \Pr[S = 5] &= \frac{2}{10} \\ \Pr[S = 3] &= \frac{2}{10} \\ \Pr[S = 0] &= \frac{5}{10} \end{aligned}$$

Thus our expected value is:

$$10 \cdot \frac{1}{10} + 5 \cdot \frac{2}{10} + 3 \cdot \frac{2}{10} + 0 \cdot \frac{5}{10} = 2\frac{3}{5}$$

Problem 27, page 230

If we assume that the coin is fair, then the expected number of heads is 5000. To figure out whether we can say that the coin is fair, we should first discover how unlikely it is that we would see a result that far (or farther) away from the expected value. That is, we want to know the chances of getting 5800 or more heads plus the chances of getting 5800 or more tails. Calculating the probabilities from the binomial distribution directly with such large numbers is prohibitive. However, we can estimate these probabilities with the DeMoivre-Laplace limit theorem. We know:

$$\begin{aligned} \Pr[4200 < X < 5800] &\approx \Pr\left[\frac{4200.5 - 5000}{\sqrt{2500}} < \frac{X - 5000}{\sqrt{2500}} < \frac{5799.5 - 5000}{\sqrt{2500}}\right] \\ &\approx \Pr\left[-16 < \frac{X - 5000}{\sqrt{2500}} < 16\right] \end{aligned}$$

The range -16 to 16 is way off Table 5.1 on page 203 of the book (which only goes to 3.49), so this probability is almost 1. This means that the coin is surely not fair.

Similar idea: From Table 5.2, the event $\left[-3.5 < \frac{X-5000}{\sqrt{2500}} < 3.5\right]$ has probability at least 0.9996 (this is the same as the event $[4825 < X < 5175]$), so an X as large as 5800 is *extreemly* unlikely for a fair coin.

Problem 33, page 231

The problem doesn't say *how long* the radio has been used; let's give a name to its age when he buys it, say, A (in years). We also need a name for the lifetime of the radio when it fails— say, T (also in years). Then the probability the used radio lasts an additional 8 years, given that it has already lasted A years, is:

$$\begin{aligned} \Pr[T > A + 8 \mid T > A] &= \frac{\Pr[T > A + 8]}{\Pr[T > A]} \\ &= \frac{e^{-(A+8)\lambda}}{e^{-(A)\lambda}} \\ &= e^{-8\lambda} \\ &= e^{-1}, \end{aligned}$$

exactly the same as the probability $\Pr[T > 8]$ that a brand-new radio would last at least eight years. This interesting feature of the exponential distribution is called "memorylessness"; the failure-time distribution does not depend on the age or past history. That is to say, the actual age of the radio doesn't affect how much longer it will work if breakdowns really do have an exponential distribution. No other distribution has this property.

Problem 38, page 231

The quadratic formula tells us that the roots of this equation are:

$$\frac{-4Y \pm \sqrt{16Y^2 - 16(Y+2)}}{8}$$

These are both real whenever the expression under the square root is non-negative, that is, when $0 \leq Y^2 - Y - 2 = (Y-2)(Y+1)$. Thus a negative discriminant occurs only when $-1 < Y < 2$. For uniform $Y \sim \mathsf{Un}(0,5)$ we know this probability is $2/5$. Thus the probability of both roots being real is $3/5$.

Exercise 25, page 234

By definition, if $X \sim \mathsf{Be}(a,b)$ then:

$$E[X] = \int_0^1 x \cdot \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} dx$$

Combining x and x^{a-1} and using 6.3 give us:

$$\begin{aligned}
E[X] &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^a (1-x)^{b-1} dx \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot B(a+1,b) \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)} \cdot \frac{a \cdot \Gamma(a)}{(a+b) \cdot \Gamma(a+b)} \\
&= \frac{a}{a+b}
\end{aligned}$$

For the variance, we will first calculate $E[X^2]$:

$$\begin{aligned}
E[X^2] &= \int_0^1 x^2 \cdot \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} dx \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+1}(1-x)^{b-1} dx \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)} \cdot \frac{(a+1)(a)\Gamma(a)}{(a+b+1)(a+b)\Gamma(a+b)} \\
&= \frac{a(a+1)}{(a+b)(a+b+1)}
\end{aligned}$$

We just calculated $E[X]$, so we can write the variance as:

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\
&= \frac{a(a+1)}{(a+b)(a+b+1)} - (\frac{a}{a+b})^2 \\
&= \frac{a(a+1)(a+b)}{(a+b)^2(a+b+1)} - \frac{a^2(a+b+1)}{(a+b)^2(a+b+1)} \\
&= \frac{ab}{(a+b)^2(a+b+1)}
\end{aligned}$$

Exercise 28, page 234

We know from the properties of cumulative distribution functions that $F_X(x)$ is nondecreasing. It may not be *strictly* monotone (Why not? Can you think of an example?) like the function $g(x)$ in Theorem 7.1, so we have to be a tiny bit more careful.

For any $0 < y < 1$ there is at least one $x_y \in (-\infty, \infty)$ such that $F_X(x_y) = y$, since $F_X(x) \to 0$ as $x \to -\infty$, $F_X(x) \to 1$ as $x \to +\infty$, and $F_X(x)$ has no jumps; this is one form of the "intermediate value theorem" from calculus class. Then, for $0 < y < 1$,

$$\begin{aligned}
F_Y(y) &= P[F(X) \le y] \\
&= P[F(X) \le F(x_y)] \\
&= P[X \le x_y] \\
&= F(x_y) \\
&= y
\end{aligned}$$

We know that $f_Y(y) = \frac{d}{dy} F_Y(y) = 1$ for $0 < y < 1$; evidently $f_Y(y) = 0$ for y outside that range (why?), so $Y \sim \mathsf{Un}(0,1)$ as claimed.

Another Problem

1)

$$\begin{aligned} P[X > 20] &= \int_{20}^{\infty} \frac{200}{x^3} dx \\ &= -100x^{-2}|_{x=20}^{\infty} \\ &= \frac{1}{4} \end{aligned}$$

2) For $x < 10$, $F(x) = 0$. If $x \geq 10$, we know:

$$F(x) = \int_{10}^{x} \frac{200}{y^3} dy = 1 - 100x^{-2}$$

3) By definition, the hazard function, $\lambda = f(x)/(1-F(x))$. For this function, that is $2/x$. This is a decreasing function, so these electronic devices become increasingly reliable as they age. Can you think of any possible explanation? This phenomenon *does* really happen!