Sta 104 Homework 6 solutions

September 30, 2003

Problem 9 p.290 (6 points)

(a) $\int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = 1$

(b) $f_X(x) = \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}(2x^2 + x)$, for $0 < x < 1$, and 0 otherwise.

(c) $P(X > Y) = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = \frac{15}{56}$

(d) $P(Y > 1/2|X < 1/2) = \frac{P(Y>1/2, X<1/2)}{P(X<1/2)} = \frac{\int_{1/2}^2 \int_0^{1/2}\left(x^2+\frac{xy}{2}\right)dxdy}{\int_0^{1/2}(2x^2+x)dx} = \frac{69}{80}$

(e) $E[X] = \int_0^1 \frac{6}{7}(2x^3 + x^2)dx = \frac{5}{7}$

(f) $E[Y] = \int_0^2 \frac{6}{7}\left(\frac{y}{3} + \frac{y^2}{4}\right) dy = \frac{8}{7}$

Problem 13 p.291

Let $M \sim \mathcal{U}_{[15,45]}$ be the time the man arrives, and $W \sim \mathcal{U}_{[0,60]}$ the time the woman arrives. Then the chance that the first to arrive waits at most five minutes is:

$$\begin{aligned}
P(\text{wait time} < 5\text{ min}) &= p(M - W < 5|M > W) + P(W - M < 5|W > M) \\
&= \int_{15}^{45}\int_m^{m+5} \frac{1}{30}\cdot\frac{1}{60}dwdm + \int_0^{60}\int_w^{w+5}\frac{1}{30}\cdot\frac{1}{60}dmdw \\
&= \frac{1}{12} + \frac{1}{6} = \frac{1}{4}
\end{aligned}$$

Part 2:

$$\begin{aligned}
P(M < W) &= 1 - \int_{15}^{45}\int_0^m \frac{1}{60}\cdot\frac{1}{30}dwdm \\
&= 1 - \int_{15}^{45} m\cdot\frac{1}{60}\cdot\frac{1}{30}dm \\
&= 0.5
\end{aligned}$$

One might also solve this part by conditioning on the arrival time of the woman. If she arrives in the first 15 minutes, she will be first. If she arrives in the last 15 minutes, she will be second. Otherwise, she has an equal chance of arriving first and second. Thus the total chance is 50%.

Problem 16 p291

a) $A = \cup A_i$

b) yes

c) Fixing i, each point is equally likely to be in the semicircle starting at A_i as not. Thus, $P(A_i) = (1/2)^{n-1}$. There are n such points, so $P(A) = n \cdot (1/2)^{n-1}$.

Problem 24 p292

a) $p_0^{n-1} \cdot (1 - p_0)$

b) This is $P(X = j | X \neq 0) = p_j/(1 - p_0)$

c) For this to happen, we need $n-1$ zeros followed by j. Thus, this probability is $p^{n-1} \cdot p_j = p_0^{n-1} \cdot (1 - p_0) \cdot p_j/(1 - p_0) = P(N = n) \cdot P(X = j)$.

Problem 32 p293

We assume that sales from week to week are independent.

a) This means that the gross sales from two weeks is a sum of two single week random variables. Therefore it is normal with mean \$4400 and standard deviation $\approx$ \$325.27. $z = (5000 - 4400)/325.27 \approx 1.8446$ We find that this corresponds to a probability of 0.9671. We are asked for the opposite event, so the solution is $1 - 0.9671 = .0329$.

b) $P(\text{sales} < 2000 \text{ in 1 week}) = P(z > (2000 - 2200/230)) \approx .8078$. Thus the probability of exceeding 2000 in two of three weeks is $0.8078^3 + 3 \cdot 0.8078^2 \cdot 0.1922 \approx .903$

Problem 50 p295

Let X_1 be the first pull from our distribution, and X_2 be the second pull. First notice that both X_1 and X_2 are restricted to the interval $[0, 1]$. Thus, the pair of variables define points in the unit square in the first quadrant. To find the distribution function of the range at $Z = |X_1 - X_2|$ we want to look in the region in which the difference is less than z. This occurs between the lines of slope 1 and intercepts z and $-z$. Also notice that the joint density is symmetric about the line y=x. Thus, we may integrate the joint density in the triangle with vertices $(z,0)$, $(1,0)$, and $(1,z)$, double what we get, and subtract this from zero. For $z \in [0,1]$ we have:

$$\begin{aligned} F_Z(z) &= 1 - 2\int_z^1 \int_0^{1-z} 2x \cdot 2y dy dx \\ &= 1 - 2[(1-z)^2 - z^2 \cdot (1-z)^2] \\ &= 1 - 2(1 - 2z + z^2 - z^2 + 2z^3 - z^4) \\ &= -1 + 4z - 4z^3 + 2z^4 \end{aligned}$$

Problem 54 p295

$$J = \begin{vmatrix} y & x \\ 1/y & -(x/y^2) \end{vmatrix} = -2\frac{x}{y}$$

Notice that $uv = x^2$ and $u/v = y^2$. Thus:

$$f_{U,V}(u,v) = \frac{1}{2vu^2} \text{ where } u \geq 1 \text{ and } 1/u \leq v \leq u$$

To find the marginal densities, we integrate this with respect to the variable we want to get rid of.

$$\begin{aligned} f_U(u) &= \int_{1/u}^{u} \frac{1}{2u^2} \cdot v^{-1} dv \\ &= -\frac{1}{2u^2} log(v)\Big|_{v=1/u}^{u} \\ &= \frac{1}{u^2} log(u) \\ f_V(v) &= \int_{1/v}^{\infty} \frac{1}{2vu^2} du \text{ for } 0 < v < 1 \\ &= \frac{1}{2v}(u^{-1})\Big|_{u=1/v}^{\infty} = \frac{1}{2} \\ f_V(v) &= \int_{v}^{\infty} \frac{1}{2vu^2} du \text{ for } v > 1 \\ &= \frac{1}{2v}(u^{-1})\Big|_{u=v}^{\infty} = \frac{1}{2v^2} \end{aligned}$$

Exercise 8 p296

a)

$$\begin{aligned}
P(w \leq t) &= 1 - P(w > t) \\
&= 1 - P(x > t, y > t) \\
&= 1 - (1 - F_x(t))(1 - F_y(t)) \text{ (this is the one we'll use for part b)} \\
&= F_x(t) + F_y(t) - F_x(t)F_y(t)
\end{aligned}$$

b)

$$\begin{aligned}
f_w(t) &= \frac{d}{dt} F_W(t) \\
&= f_X(t)(1 - F_Y(t)) + (1 - F_X(t))f_Y(t) \\
\lambda_W &= \frac{f_W}{1 - F_W} \\
&= \frac{f_X(t)(1 - F_Y(t)) + (1 - F_X(t))f_Y(t)}{(1 - F_x(t))(1 - F_y(t))} \\
&= \frac{f_X(t)}{1 - F_X(t)} + \frac{f_Y(t)}{1 - F_Y(t)} \\
&= \lambda_X(t) + \lambda_Y(t)
\end{aligned}$$

Another Problem

We Assume that where one coin lands doesn't have any bearing on where the others land, that is we assume that where these three coins land are independent and identically distributed. The center of each coin is a total distance from the center of the can between $[0, 2]$. So $P(R \leq r) = \frac{\pi r^2}{\pi 2^2} = \frac{r^2}{4}$, for $0 < r < 2$. Since these are independent, $F_{R_{(3)}}(r) = P(R_1 \leq r, R_2 \leq r, R_3 \leq r) = P(R_1 \leq r)P(R_2 \leq r)P(R_3 \leq r) = \frac{r^6}{64}, 0 < r < 2$. Then $f_{R_{(3)}}(r) = \frac{3r^5}{32}, 0 < r < 2$.