Sta 104 Homework 7 solutions

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Problem 5 p.380

The joint density of the point (X, Y) at which the accident occurs is

$$f(x,y) = \begin{cases} \frac{1}{9}, & -3/2 < x, y < 3/2 \\ 0, & \text{otherwise} \end{cases}$$

$$= f(x)f(y)$$

where

$$f(a) = \begin{cases} \frac{1}{3}, & -3/2 < a < 3/2 \\ 0, & \text{otherwise} \end{cases}$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$. Therefore,

$$E(|X| + |Y|) = \frac{2}{3}\int_{-3/2}^{3/2} |x| dx = \frac{4}{3}\int_0^{3/2} x dx = 3/2$$

Problem 12 p.381

Let $m = n$ (for those who use the 5^{th} edition).

(a) Let X_i equal 1 if the person in position i is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \frac{1}{2}\frac{n}{2n-1}, & \text{if } i = 1, 2n \\ \frac{1}{2}\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore, $E\left[\sum_{i=1}^{2n} X_i\right] = \sum_{i=1}^{2n} E[X_i] = \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) = \frac{3n^2-n}{4n-2} = \left(\frac{mn(2n+m-1)}{(m+n)(m+n-1)} \text{ for } 5_{th} \text{ edition}\right)$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result:

$$n\left[1 - \frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2} = \left(\frac{2mn(n-1) + nm(m-1)}{(m+n)(m+n-1)} \text{ for } 5_{th} \text{ edition}\right)$$

where the right side equality assumes that $n > 1$.

Problem 14 p.381

The number of stages is a negative binomial random variable with parameters m and $1 - p$. Hence, its expected value is $m/(1 - p)$.

Problem 19 p.382

(a) E[time of first type 1 catch] $= \frac{1-p_1}{p_1}$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1, & \text{if a type } j \text{ is caught before a type 1} \\ 0, & \text{otherwise} \end{cases}$$

Then $E\left[\sum_{j\neq 1} X_j\right] = \sum_{j\neq 1} E[X_j] = \sum_{j\neq 1} P(\text{type } j \text{ before type 1}) = \sum_{j\neq 1} P_j/(P_j + P_1)$.

Where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give

$P(\text{type } j \text{ before type 1}) = P(j \mid j \text{ or } i) = \frac{P_j}{P_j+P_1}$

Problem 42 p.384

(a) Let

$$X_i = \begin{cases} 1, & \text{if pair } i \text{ consists of a man and a woman} \\ 0, & \text{otherwise} \end{cases}$$

$E[X_i] = P(X_i = 1) = \frac{10}{19}$
$= E[X_iX_j] = P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1)$
$= \frac{10}{19}\frac{9}{17}, i \neq j$

$E\left[\sum_1^{10} X_i\right] = \frac{100}{19} = 5.263$

$Var\left[\sum_1^{10} X_i\right] = 10\frac{10}{19}\left(1 - \frac{10}{19}\right) + 10\cdot 9\left(\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right) = \frac{900}{19^2}\frac{18}{17} = \frac{16200}{6137} = 2.639$

(b)

$$X_i = \begin{cases} 1, & \text{if pair } i \text{ consists of a married couple} \\ 0, & \text{otherwise} \end{cases}$$

$E[X_i]n = \frac{1}{19}, E[X_iX_j] = P(X_i = 1)P(X_j = 1|X_i = 1) = \frac{1}{19}\frac{1}{17}, i \neq j$

$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$

$Var\left[\sum_1^{10} X_i\right] = 10\frac{1}{19}\frac{15}{19} + 10\cdot 9\left(\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right) = \frac{180}{19^2}\frac{18}{17} = \frac{3240}{6137} = 0.528$

Problem 51 p.385

$f_{X|Y}(x|y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y dx} = \frac{1}{y}, 0 < x < y$
$E[X^3|Y = y] = \int_0^y x^3\frac{1}{y}dx = \frac{y^3}{4}$

Exercise 1 p.389

$\frac{d}{da}E[(X-a)^2] = \frac{d}{da}[E(X^2) - 2aE(X) + a^2] = -2E(X) + 2a$. Setting this equal to zero gives a solution of $a = E(X)$ as either a maximum or a minimum. We check the second derivative to determin which, $\frac{d^2}{da^2}E[(X-a)^2] = \frac{d}{da}[-2E(X) + 2a] = 2 > 0$, which implies that we have a minimum.

Another Problem

For the first question, we note that if each person is assigned each plan with probability .5, then the expected number of minutes purchased per person is 650 and the expected cost per person is \$44. There are 700 employees, so that is $650 \times 700 = 455000$ minutes purchased at a cost of $700 \times \$44 = \$30,800$. Our expected cost will then be:

$$\begin{aligned} E(C) &= \int_{325000}^{455000} 30800 \cdot \frac{1}{150000} dy + \int_{455000}^{475000} [30800 + .25(y - 455000)] \cdot \frac{1}{150000} dy \\ &= 26693.3 + \left(\frac{y^2}{1200000} - .553y \right) \Bigg|_{y=455000}^{475000} \\ &= 26693.3 + 4440 = \$31,133.33 \end{aligned}$$

Now we want to minimize cost. First we need to know what our cost will be. We will let Y be the random variable signifying the number of minutes used in the month, x will be the number of 400 minute plans purchased, and C will be the cost. Also, we will use $f_Y(y)$ to signify the density of Y. In this case it is uniform, but to provide a partial answer to the last part, I will leave it as general as possible for now. Then our cost is:

$$C = \begin{cases} \text{32x+56(700-x)} & \text{when} \quad Y < 400x + 800(700 - x) \\ \text{32x+56(700-x) + .25(Y-400x+800(700-x))} & \text{when} \quad Y > 400x + 800(700 - x) \end{cases}$$

Our expected cost is then

$$\begin{aligned} E(C) &= \int_{-\infty}^{400x+800(700-x)} [32x + 56(700 - x)] \cdot f_Y(y) dy \\ &+ \int_{400x+800(700-x)}^{\infty} [32x + 56(700 - x) + .25(y - 400x - 800(700 - x))] \cdot f_Y(y) dy \\ &= \int_{-\infty}^{\infty} [32x + 56(700 - x)] \cdot f_Y(y) dy \\ &+ \int_{400x+800(700-x)}^{\infty} [.25(y - 400x - 800(700 - x))] \cdot f_Y(y) dy \\ &= 32x + 56(700 - x) + \int_{400x+800(700-x)}^{\infty} [.25y + 100x - 140000] \cdot f_Y(y) dy \end{aligned}$$

Now we will go back to the uniform distribution.

$$\begin{aligned} \frac{\partial}{\partial x} E(C) &= \frac{\partial}{\partial x} \left(-24x + 39200 + \int_{400x+800(700-x)}^{475000} \frac{.25y + 100x - 140000}{150000} dy \right) \\ &= \frac{\partial}{\partial x} \left(-24x + 39200 + \left[\frac{y^2}{8} + (100x - 140000)y \right]_{y=400x+800(700-x)}^{475000} \right) \\ &= -24 + 316.67 + .2667x - 373.333 \end{aligned}$$

Setting this equal to zero leaves us with $x \approx 302$ which corresponds to a total purchased minutes of

439,200. Putting this into our expected cost equation, we get:

$$\begin{aligned}
E(C) &= \int_{325000}^{439200} 22288 \cdot \frac{1}{150000} dy + \int_{439200}^{475000} [22288 + .25(y - 439200)] \cdot \frac{1}{150000} dy \\
&= 16968.60 + \left(\frac{y^2}{1200000} - .583y \right) \Bigg|_{y=439200}^{475000} \\
&= 16968.60 + 6548.6 = \$23,517.20
\end{aligned}$$