Sta 104 Homework 8 solutions

November 12, 2003

Problem 40, page 384

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{1}{y} e^{-(y+x/y)} dx \\ &= -e^{-(y+x/y)}\Big|_{x=0}^{\infty} \\ &= e^{-y} \text{ for } y > 0 \text{ and zero otherwise.} \end{aligned}$$

This is exponential with parameter 1, so it has an expected value of 1.

We will use the conditional expectation to calculate $E[X]$. First we need to find $f(X|Y)$.

$$\begin{aligned} f(x|y) &= \frac{(1/y)e^{-(y+x/y)}}{e^{-y}} \\ &= \frac{1}{y} e^{-(x/y)} \text{ for } x, y > 0 \text{ and zero otherwise.} \end{aligned}$$

This is exponential with mean y, which leads us to:

$$E[X] = E[E[X|Y]] = E[Y] = 1$$

Finally,

$$\begin{aligned} Cov(X,Y) &= E[(X-1)(Y-1)] \\ &= E[XY] - E[X] - E[Y] + 1 \\ &= E[XY] - 1 \\ &= E[E[XY|Y]] - 1 \\ &= E[Y \cdot E[X|Y]] - 1 \\ &= E[Y^2] - 1 \end{aligned}$$

We know that Y is exponential with mean and variance equal to 1, so $Var[Y] = E[Y^2] - E[Y]^2$ leads us to $E[Y^2] = 2$.

Thus we see that $Cov(X,Y) = 1$.

Problem 70, page 388

a)

$$P(\text{heads}) = \int_0^1 p dp = 1/2$$

b)

$$P(\text{heads}) = \int_0^1 p^2 dp = 1/3$$

Exercise 4, page 390

The first three terms of the Taylor series expansion of g around μ lead us to:

$$g(x) \approx g(\mu) + (x - \mu) \cdot g'(\mu) + (x - \mu)^2 \cdot g''(\mu)/2$$

Taking expectations of both sides leads us to our answer:

$$\begin{aligned} E[g(x)] &\approx g(\mu) + E[(x - \mu)] \cdot g'(\mu) + E[(x - \mu)^2] \cdot g''(\mu)/2 \\ &= g(\mu) + \sigma^2 g''(\mu)/2 \end{aligned}$$

Exercise 19, page 392

$$\begin{aligned} Cov[X + Y, X - Y] &= Cov[X, X] + Cov[X, -Y] + Cov[Y, X] + Cov[Y, -Y] \\ &= Var(X) - Cov[X, Y] + Cov[X, Y] - Var(Y) \\ &= Var(X) - Var(Y) = 0 \end{aligned}$$

Exercise 48, page 396

$$\begin{aligned} \phi_Y(t) &= E[e^{tY}] \\ &= E[e^{t(aX+b)}] \\ &= x^{tb} \cdot E[e^{atX}] \\ &= x^{tb} \cdot \phi_X(at) \end{aligned}$$

Exercise 54, page 397

$$\begin{aligned} Cov(Z, Z^2) &= E[(Z - \mu)(Z^2 - E(Z^2))] \\ &= E[Z^3 - Z^2\mu - z(\mu^2 + \sigma^2) + \mu(\mu^2 + \sigma^2)] \\ &= E[Z^3] - \mu(\mu^2 + \sigma^2) - \mu(\mu^2 + \sigma^2) + \mu(\mu^2 + \sigma^2) \\ &= E[Z^3] - \mu(\mu^2 + \sigma^2) \end{aligned}$$

Notice that, since the standard normal distribution is symmetric, $E[z^3] = E[-z^3] = -E[z^3]$. The only way this could be is if $E[z^3] = 0$. (For every point that the function is positive to the right of zero, there is a point that is negative to the left of zero.)

Another Problem

Clearly, $E[X_1] = 1$ as stated. Now we need to know $E[X_2]$, $E[X_3]$, and $E[X_4]$. For each of the next dollars handed out, the probability it going to someone from a different class is $3/4$. This is a geometric random variable with parameter $p = 3/4$. Thus its expected value is $4/3$. Similarly, X_3 and X_4 are geometric with parameters $1/2$ and $1/4$ respectively. Thus the expected value for the number of dollars paid out is $1 + 4/3 + 2 + 4 = 25/3$.

Part two asks for the probability that \$4 is paid out (which we will call D_4) plus the probability that \$5 is paid (which we call D_5).

$$\begin{aligned}
P(D_4) &= 1 \cdot 3/4 \cdot 1/2 \cdot 1/4 = 3/32 \\
P(D_5) &= 1 \cdot (1/4)(3/4) \cdot 1/2 \cdot 1/4 \\
&+ 1 \cdot 3/4 \cdot (1/2)(1/2) \cdot 1/4 \\
&+ 1 \cdot 3/4 \cdot 1/2 \cdot (3/4)(1/4) \\
&= 3/128 + 3/64 + 3/128 = 3/32
\end{aligned}$$