Sta 104 Homework 10 Solutions

December 5, 2001

All problems are worth 4 points.

Problem 4

(a) $P\left(\sum_{i=1}^{20} X_i > 15\right) \leq 20/15 = 4/3$

(b) $P\left(\sum_{i=1}^{20} X_i > 15\right) = P\left(\sum_{i=1}^{20} X_i > 15.5\right)$
$\approx P\left(Z > \frac{15.5-20}{\sqrt{20}}\right)$
$= P(Z > -1.006)$
$\approx .8428$

Problem 5

Letting X_i denote the i^{th} roundoff error it follows that $\mathrm{E}\left[\sum_{i=1}^{50} X_i\right] = 0$, $\mathrm{Var}\left[\sum_{i=1}^{50} X_i\right] = 50\,\mathrm{Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform $(0, 1)$ and so $\mathrm{Var}(X) = Var(.5 + X) = 1/12$. Hence,

$P\left(|\sum X_i| > 3\right) \approx P(|Z| > 3\sqrt{12/50})$ by the central limit theorem

$= 2P(Z > 1.47) = 0.1416.$

Problem 6

If X_i is the outcome of the i^{th} roll then $\mathrm{E}[X_i] = 7/2$, $\mathrm{Var}[X_i] = 35/12$ and so $P\left(\sum_{i=1}^{79} X_i \leq 300\right) = P\left(\sum_{i=1}^{79} X_i \leq 300.5\right) \approx P\left(Z \leq \frac{300.5-79(7/2)}{\sqrt{79 \cdot 35/12}}\right) = P(Z \leq 1.58) = .9429$

Problem 10

If W_n is the total weight of n cars and A is the amount of weight that the bridge can withstand then $W_n - A$ is Normal with mean $3n - 400$ and variance $.09n + 1600$. Hence , the probability of structural damage is

$$P(W_n - A \geq 0) \approx P\left(Z \geq (400 - 3n)/\sqrt{.09n + 1600}\right)$$

Since $P(Z \geq 1.28) = .1$ the probability of damage will exceed .1 when n is such that

$$400 - 3n \leq 1.28\sqrt{.09n + 1600}$$

This happens whenever $n \geq 117$.

Problem 11

By backward substitution, the stock price in ten days can be modeled as:

$$\begin{aligned} Y_{t+2} &= (Y_t + X_{t+1}) + X_{t+2} \\ Y_{t+3} &= (Y_t + X_{t+1}) + X_{t+2} + X_{t+2} \\ &\dots \\ Y_{t+10} &= Y_t + \sum_{i=1}^{10} X_{t+i} \end{aligned}$$

Since the X_t are $i.i.d$ we can write:

$$\begin{aligned} \mathrm{E}(\sum_{i=1}^{10} X_{t+i}) &= 0 \\ \mathrm{Var}(\sum_{i=1}^{10} X_{t+i}) &= 10 \end{aligned}$$

and by setting $Y_t = 100$ and $\sum_{i=1}^{10} X_{t+i} = X$ we have:

$$Y_{t+10} = 100 + X$$

so that the probability statement:

$$P(Y_{t+10} > 105)$$

is equivalent to:

$$P(X > 5)$$

and by the central limit theorem:

$$\begin{aligned} P(\frac{X}{\sqrt{10}} > \frac{5}{\sqrt{10}}) &\approx 1 - \Phi(\frac{5}{\sqrt{10}}) \\ &= 1 - \Phi(1.581139) \\ &\approx 0.0572 \end{aligned}$$

Problem 13

(a) $P(\bar{X} > 80) = P\left(\frac{\bar{X}-74}{14/5} > 15/7\right) \approx P(Z > 2.14) \approx .0162$

(b) $P(\bar{Y} > 80) = P\left(\frac{\bar{Y}-74}{14/8} > 24/7\right) \approx P(Z > 3.43) \approx .0003$

(c) Using that $\mathrm{SD}(\bar{Y} - \bar{X}) = \sqrt{196/64 + 196/25} \approx 3.30$ we have

$$P(\bar{Y} - \bar{X} > 2.2) = P\left((\bar{Y} - \bar{X})/3.30 > 2.2/3.3\right) \approx P(Z > .67) \approx .2514$$

(d) same as in (c)

Problem 15

$P\left(\sum_{i=1}^{10,000} X_i > 2,700,000\right) \approx P\left(Z \geq (2,700,000 - 2,400,000)/(800 \cdot 100)\right)$
$= P(Z \geq 3.75) \approx 0$

Problem 18

Let Y_i denote the additinoal number of fish that need to be caught to obtain a new type when there are at present i distinct types. Then Y_i is geometric with parameter $\frac{4-i}{4}$.

$$\mathrm{E}[Y] = \mathrm{E}\left[\sum_{i=0}^{3} Y_i\right] = 1 + 4/3 + 4/2 + 4 = 25/3$$

$$\mathrm{Var}[Y] = \mathrm{Var}\left[\sum_{i=0}^{3} Y_i\right] = 4/9 + 2 + 12 = 130/9$$

(a) Hence,

$$P\left(\left|Y - \frac{25}{3}\right| > \sqrt{\frac{1300}{9}}\right) \leq 1/10$$

and so we can take $a = \frac{25-\sqrt{1300}}{3} = -3.69$, $b = \frac{25+\sqrt{1300}}{3} = 20.35$

(b)

$$P\left(Y - 25/3 > a\right) \leq \frac{130}{130 + 9a^2} = 1/10 \text{ when } a = \frac{\sqrt{1170}}{3} = 11.4$$

Hence, $P\left(Y > \frac{25+\sqrt{1170}}{3}\right) \leq .1$

Exercise 7

The approximation can be obtained by doing the following:

i) take logarithms on both sides;

ii) apply the central limit theorem.

It follows that:

$$P\left(\prod_{i=1}^{100} X_i \leq a^{100}\right) = P\left(\sum_{i=1}^{100} \log(X_i) \leq 100\log(a)\right)$$

Consider the independent and identically distributed ($i.i.d.$) random variables $\log(X_i)$. Their probability mass function can be represented as:

$$\log(X_i) \sim \begin{cases} 0 & \log(2) & \log(3) & \log(4) & \log(5) & \log(6) \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{cases}$$

with: $\mathrm{E}(X_i) = 1.096542$, $\mathrm{Var}(X_i) = 0.3659136$, $\sqrt{\mathrm{Var}(X_i)} = 0.6049079$. We apply the central limit theorem and get an acceptable approximation for the distribution of $\sum_{i=1}^{100} \log(X_i)$, so that:

$$P\left(\frac{\sum_{i=1}^{100} \log(X_i) - 109.6542}{6.049079} \leq \frac{100\log(a) - 109.6542}{6.049079}\right) \approx \Phi\left(\frac{100\log(a) - 109.6542}{6.049079}\right)$$

where $\Phi(\cdot)$ is the cumulative probability function of the standard Normal random variable. For every value of $a \in (1, 6)$ this gives us an approximation for the above probability.

Another problem

a)

$$\begin{aligned}
\lim_{n\to\infty} P\left(X_n < x\right) &= \lim_{n\to\infty} P\left(n(1 - Z_n) < x\right) \\
&= \lim_{n\to\infty} P\left(Z_n > 1 - \frac{x}{n}\right) \\
&= 1 - \lim_{n\to\infty} P\left(Z_n \leq 1 - \frac{x}{n}\right) \\
&= 1 - \lim_{n\to\infty} P\left(\max(U_1, ..., U_n) \leq 1 - \frac{x}{n}\right) \\
&= 1 - \lim_{n\to\infty} P\left(U_1 \leq 1 - \frac{x}{n}, U_2 \leq 1 - \frac{x}{n}, ..., U_n \leq 1 - \frac{x}{n}\right) \\
&= 1 - \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n \\
&= 1 - e^{-x} = F_X(x) \text{ for } x > 0
\end{aligned}$$

The above result implies that $f_X(x) = e^{-x}$ for $x > 0$ so that the limiting distribution is Exponential with parameter $\lambda = 1$

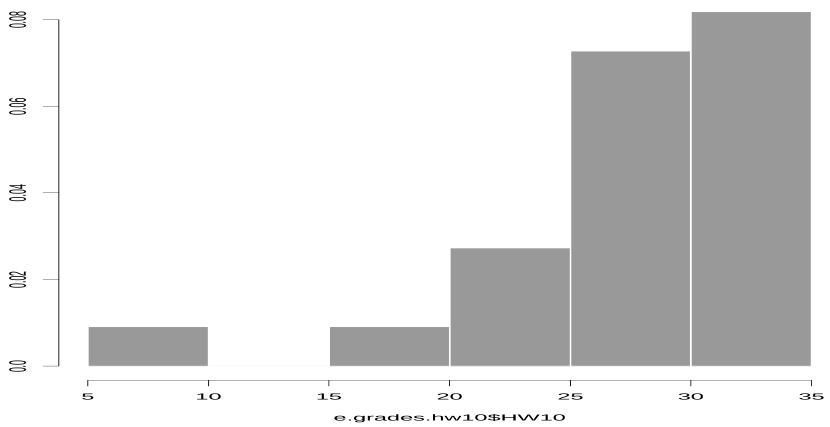
b) Since U_i is a uniform distribution on the $(0,1)$ interval, the random variable $U_i - 1/2$ is uniform over $(-1/2, 1/2)$, with mean 0 and variance $\frac{1}{12}$. Then the application of the central limit theorem yields:

$$\begin{aligned}
\lim_{n\to\infty} P(Y_n \leq y) &= \lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n}(U_i - 1/2)}{\sqrt{n}} \leq y\right) \\
&= \lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n}(U_i - 1/2)}{\sqrt{n/12}} \leq y\sqrt{12}\right) \\
&= \Phi\left(y\sqrt{12}\right)
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative probability function of the standard Normal random variable, thus Y has the Normal distributoin with mean 0 and variance $\frac{1}{12}$

Distribution for Homework 10

25 total points possible



	Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max
Raw Score	9.0	27.3	29.0	27.9	31.8	35.0
Percentage	0.23	0.68	0.73	0.70	0.79	0.88