10.0 Lesson Plan

- Discuss Quizzes/Answer Questions
- Review Probability
- Some Exercises
- Bayes' Rule

10.1 Review

Recall the addition rule for OR, where "or" means that either A or B or both occur.

$$P[A \textbf{ or } B] = P[A] + P[B] - P[A \textbf{ and } B].$$

There are two special cases. One arises when A and B are mutually exclusive, and the other arises when A and B are independent.

- If A and B are mutually exclusive, then it is impossible for both to happen, so $P[A \text{ and } B] = 0$ and the formula reduces to:

$$P[A \textbf{ or } B] = P[A] + P[B]$$

(as in the Kolmogorov axiom).

- If A and B are independent events then the occurrence of one does not affect the occurrence of the other, so $P[A \text{ and } B] = P[A] * P[B]$ and the formula reduces to:

$$P[A \textbf{ or } B] = P[A] + P[B] - P[A] * P[B].$$

Recall the multiplication rule for AND, where "and" means that both of A and B must occur.

$$P[A \textbf{ and } B] = P[A|B] * P[B] = P[B|A] * P[A].$$

Here, the conditional probability $P[A|B]$ is the probability that event A occurs, given that event B is known to occur. The mathematical definition is:

$$P[A|B] = P[A \textbf{ and } B]/P[B].$$

As before, there are two special cases that arise when the events are mutually exclusive or independent.

- If A and B are mutually exclusive, $P[A \text{ and } B] = 0$.
- If A and B are independent events then the occurrence of one does not affect the occurrence of the other, so $P[A|B] = P[A]$ and, equivalently, $P[B|A] = P[B]$. Then:

$$\begin{aligned} P[A \textbf{ and } B] &= P[A|B] * P[B] = P[A] * P[B] \\ P[A \textbf{ and } B] &= P[B|A] * P[A] = P[B] * P[A] \end{aligned}$$

Note that if A is independent of B, then B is independent of A. That is, if the occurrence of A gives no information about B, then the occurrence of B gives no information about A. This is true, but perhaps not obvious.

The binomial formula gives the probability of exactly r "successes" in n tries, where n is fixed, each try is independent, and the probability of success on each try is p.

$$P[\ \textbf{exactly}\ r\ \textbf{successes}\ ] = \begin{pmatrix} n \\ r \end{pmatrix} p^r (1-p)^{n-r}$$

where the coefficient $\begin{pmatrix} n \\ r \end{pmatrix}$ is the number of different ways that one can pick r of the outcomes from the set of n to be successes.

Formally,

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r! * (n-r)!}$$

where $n! = n * (n-1) * (n-2) * \cdots * 1$ (and $0!$ is defined to be 1).

10.2 Some Exercises

Suppose 30% of the students at Duke are direct descendants of Caligula. You go on 12 random dates. What is the probability of dating at least one descendant?

$$\begin{aligned} P[\text{ \textbf{at least one} }] &= 1 - P[\text{ \textbf{no descendants} }] \\ &= 1 - P[\text{ \textbf{first is not} }] * P[\text{ \textbf{second is not} }] * \cdots * \\ &\quad P[\text{ \textbf{twelfth is not} }] \\ &= 1 - (.7)^{12} \\ &= .986. \end{aligned}$$

This used the trick that finding the probability of “none” was easier than finding the probablity of “one or more”. Since $P[A] + P[A^c] = 1$ (recall A^c is the complement, or opposite, of A), we can subtract the probability of “none” from 1 to get the probability of “one or more”.

You could also use the binomial formula. The long way is to find

$$\begin{aligned} P[\text{ \textbf{at least one} }] &= P[\text{ \textbf{exactly one descendant} }] + \cdots + \\ &\quad P[\text{ \textbf{exactly 12 descendants} }]. \end{aligned}$$

But the trick used above applies here too:

$$\begin{aligned} P[\text{ \textbf{at least one} }] &= 1 - P[\text{ \textbf{exactly 0} }] \\ &= 1 - \binom{12}{0} (.3)^0 (1 - .3)^{12} \\ &= 1 - .7^{12}. \end{aligned}$$

What is the probability of two or fewer dates with Caligula scions?

$$
\begin{aligned}
P[\ \textbf{two or fewer}\ ] &= P[\ \textbf{exactly 0 descendants}\ ] + \\
&\quad P[\ \textbf{exactly 1 descendant}\ ] + \\
&\quad P[\ \textbf{exactly 2 descendants}\ ] \\
&= \binom{12}{0} .3^0 .7^{12} + \binom{12}{1} .3^1 .7^{11} + \binom{12}{2} .3^2 .7^{10} \\
&= 1 * .7^{12} + 12 * .3 * .7^{11} + 66 * .3^2 * .7^{10} \\
&= .00138 + .07118 + .16779 \\
&= .2528.
\end{aligned}
$$

The Birthday Problem asks "What is the probability that two or more people in a class of size n have the same birthday?"

We know that

$$P[\ \textbf{2 or more matches}\ ] = 1 - P[\ \textbf{no match}\ ].$$

If there are two people in the class, then $n = 2$ and the probability of no match is

$$P[\ \textbf{no match}\ ] = \frac{365}{365} * \frac{364}{365}.$$

If $n = 3$ then

$$P[\textbf{ no match }] = \frac{365}{365} * \frac{364}{365} * \frac{363}{365}.$$

For $n = 4$

$$P[\textbf{ no match }] = \frac{365}{365} * \frac{364}{365} * \frac{363}{365} * \frac{362}{365}.$$

The pattern continues. In general,

$$P[\textbf{ no match in } n \;] = \frac{(365)(364) * \cdots * (365 - n + 1)}{365^n}.$$

Therefore, using the complementation trick,

$$P[\text{ one or more matches in } n\ ] = 1 - \frac{(365)(364) * \cdots * (365 - n + 1)}{365^n}.$$

For $n = 26$ the probability of a match is about .6. For $n = 50$ the probability of a match is about .97

This assumes that all days of the year are equally likely to be birthdays. This is only approximately true.

Consider poker game problems. Suppose you hold a king of hearts, a queen of hearts, a jack of hearts, a six of clubs, and two of spades. You discard the six and the two, and draw replacements. What is the chance that you hold just a pair of jacks?

For this to happen, one of your draws must be a jack, and the other must not be a king, queen or jack. If the jack is drawn first, then the chance is $\frac{3}{47} * \frac{46-3-3-2}{46} = .05273$.
Now assume that the jack is drawn last. Then the chance of getting just a pair of jacks is $\frac{47-3-3-3}{47} * \frac{3}{46} = .05273$.

So the probability of two jacks is .10546.

10.3 Bayes' Rule

The simple version of Bayes' Rule is just an application of the definition of conditional probability.

Let $A_1, \ldots, A_n$ be mutually exclusive and suppose that $P[A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n] = 1$. Then

$$P[A_1|B] = \frac{P[B|A_1] * P[A_1]}{\sum_{i=1}^{n} P[B|A_i] * P[A_i]}.$$

The point of Bayes' Rule is to reverse the order of the conditioning. One finds the probability of A_1 given B in terms of the probabilities of B given the A_i.

From the standpoint of the Bayesian definition of probability, suppose one's initial guess is that the coin is fair. Then one observes 10 heads in a row. How should one's belief about the probability of heads change?

Bayes Rule says how your belief about the probability of fairness should change. The probability of fairness given that you observed 10 straight heads can be calculated in terms of the probability of getting 10 straight heads given that the coin is fair.

A coin has probability .4 of coming up heads. If you toss a heads, you draw a marble from an urn that has 70% red balls, 30% yellow. If you toss tails, you draw from an urn that is 40% red and 60% yellow. You show me a red marble—what is the probability that you threw a heads?

Set B={red}, A_1={heads}, A_2={tails}.

$$P[\ \textbf{heads} \mid \textbf{red}\ ] =$$

$$\frac{P[\ \textbf{red} \mid \textbf{heads}\ ] * P[\ \textbf{heads}\ ]}{P[\ \textbf{red} \mid \textbf{heads}\ ] * P[\ \textbf{heads}\ ] + P[\ \textbf{red} \mid \textbf{tails}\ ] * P[\ \textbf{tails}\ ]}$$

$$= (.7) * (.4)/[(.7) * (.4) + (.4) * (.6)]$$

$$= .5385.$$

ELISA is a test for AIDS.

- If a person has AIDS, ELISA has probability .997 of signalling.
- If a person does not have AIDS, then ELISA does not signal with probability .985.
- About .32% of the U.S. population has AIDS

Suppose you get an AIDS test (e.g., as part of a marriage license). Your test comes back positive. What is the chance that you have AIDS?

We can use Bayes Rule. Let B={positive test}, $A_1 = \{\text{have AIDS}\}$, and let $A_2 = \{\text{do not have AIDS}\}$.

$$P[\ \mathbf{AIDS} \mid \mathbf{pos}\ ] =$$

$$\frac{P[\ \mathbf{pos} \mid \mathbf{AIDS}\ ] * P[\ \mathbf{AIDS}\ ]}{P[\ \mathbf{pos} \mid \mathbf{AIDS}\ ] * P[\ \mathbf{AIDS}\ ] + P[\ \mathbf{pos} \mid \mathbf{OK}\ ] * P[\ \mathbf{OK}\ ]}$$

$$= (.997) * (.0032)/[(.997) * (.0032) + (1 - .985) * (1 - .0032)]$$

$$= .1758.$$

So even though the test is positive, you are still unlikely to have AIDS. This is because the background rate of AIDS is quite low.