8.0 Lesson Plan

- Discuss Quizzes/Answer Questions
- Probability Definitions
- Drawing from a Box
- Conditional Probability, Independence
- The Addition Rule

8.1 Probability Definitions

There are two ways to define the probability of an event.

A frequentist says that the probability of event A (or $P[A]$) is the proportion of times that A occurs in a infinite sequence of separate tries.

Thus

$$P[A] = \lim_{n \to \infty} \frac{\textbf{\# times A happens}}{n}.$$

A Bayesian can pick whatever number they prefer for $P[A]$, based on their own personal experience and intuition, provided that number is consistent with all of the other probabilities they choose.

Thus a frequentist would define $P[$ heads in a coin toss $]$ as the limit of the proportion of heads in n tosses.

A Bayesian might declare their personal belief about the coin based on symmetry, or knowledge of the integrity of the coin's owner, or divine inspiration. The Bayesian's view must:

- conform to all other personal opinions
- change as new data arise according to Bayes' Rule.

Whether one is frequentist or Bayesian, all probabilities must obey Kolmogorov's Axioms:

- $0 \leq P[A] \leq 1$
- $P[$ some possible event happens $] = 1$ (one of the possible outcomes must occur).
- If A and B are incompatible events, then $P[A \text{ or } B] = P[A] + P[B]$.

Two events A and B are incompatible if it is impossible for both A and B to happen at the same time; e.g., you cannot throw a head and tail on the same toss.

From these three rules, everything else can be derived.

Kolmogorov (1903-1987) was one of the greatest mathematicians of the 20th century. This axiomatization was a trivial accomplishment.

8.2 Drawing From A Box

For the Selective Service lottery, put all birthdates for 2005 in a box. Mix thoroughly and draw one out.

- What is the probability that the date is in December?
- What is the probability that the date is not in December?
- What is the probability that the date is in January or February?

Why is the assumption that the numbers be thoroughly mixed in the box important?

Suppose the officer draws two dates from the box (without replacement). What is the probability that both are in December?

To answer this we must count all the ways in which one can make two draws. There are

$$365 * 364 = 132860$$

ways to draw two different dates. Of these,

$$31 * 30 = 930$$

are both in December. So

$$P[\ \textbf{2 December dates}\ ] = 930/132860 = .0069998.$$

You can see why people became suspicious in 1970.

A shorter way to see this is to note that there are 31/365 ways to draw December on the first draw, and after removing that date there are 30/364 chances for December on the second draw. Then

$$P[\ \textbf{2 December dates}\ ] = \frac{31}{365}\frac{30}{364} = .0069998.$$

But it isn't obvious why this multiplication rule should work...

These arguments all use frequentist definitions of probability. We are counting the number of different ways in which the event can happen, and we assume all ways are equally likely, and thus repeated trials should, in the long run, tend towards the proportion of ways in which the event can occur.

This method is not very useful in answering "What is the probability of life on Mars?" or "What is the chance that Abelard will get married?"

8.3 Conditional Probability

The probability of event A, after event B has occurred, is called the conditional probability of A given B. We write this as $P[A|B]$.

To calculate this, we define

$$P[A|B] = \frac{P[A \textbf{ and } B]}{P[B]}.$$

This justifies the previous use of multiplication, since:

$$P[B] * P[A|B] = P[A \textbf{ and } B].$$

You make two draws, without replacement, from a standard 52 card deck. What is the probability that the first card is a king and the second card is a queen?

Define:

- A = queen on the second draw
- B = king on the first draw.

Then

$$\begin{aligned} P[A \textbf{ and } B] &= P[B] * P[A|B] \\ &= \frac{4}{52} * \frac{4}{51} \\ &= .006033. \end{aligned}$$

Using conditional probability often makes calculations easy.

You roll a die. What is the probability of a six, given that the result is greater than or equal to 3?

Define:

- A = you roll a six
- B = you roll a three or larger.

Then

$$\begin{aligned} P[A|B] &= \frac{P[A \textbf{ and } B]}{P[B]} \\ &= (1/6)/(4/6) \\ &= .25 \end{aligned}$$

So the conditioning does not have to be sequential in time. Here we are asking about a subset of event B, a three or better.

Roll a die twice. What is the probability of a six on the second throw if the first throw was a six?

Define: A = second throw a six; B = first throw a six. Then

$$\begin{aligned} P[A|B] &= \frac{P[A \textbf{ and } B]}{P[B]} \\ &= (1/36)/(1/6) \\ &= 1/6. \end{aligned}$$

If $P[A|B] = P[A]$, then A is independent of B.

When events are independent, the occurrence of one does not affect the chance of the occurrence of the other.

Using the conditional probability rule, one can see that if A and B are independent, then

$$P[A \textbf{ and } B] = P[A] * P[B].$$

If A is independent of B, then B is independent of A. How can one prove this?

Two coin tosses are thought to be independent, because the outcome on the first toss does not affect the outcome on the second. (But really, this is just an assumption...)

8.4 The Addition Rule

Suppose you wanted to know the probability that event A and/or event B occur?

The formula for this is

$$P[A \textbf{ or } B] = P[A] + P[B] - P[A \textbf{ and } B].$$

If A and B are incompatible (or mutually exclusive), then what is $P[A \text{ and } B]$?

If A and B are independent, then what is $P[A \text{ and } B]$?

Example 1:

Consider a single draw from a deck of cards. Let A be the event that the card is a king, and B be the event that the card is a queen.

What is the probability of A or B? Clearly

$$\begin{aligned} P[A \textbf{ or } B] &= P[A] + P[B] - P[A \textbf{ and } B] \\ &= 4/52 + 4/52 - 0/52 \\ &= 2/13. \end{aligned}$$

Example 2:

Consider a single draw from a deck of cards. Let A be the event that the card is a king, and B be the event that the card is red.

What is the probability of A or B? Clearly

$$\begin{aligned} P[A \textbf{ or } B] &= P[A] + P[B] - P[A \textbf{ and } B] \\ &= 4/52 + 26/52 - 2/52 \\ &= 7/13. \end{aligned}$$

Example 3:

Consider two draws from a deck of cards with replacement. Let A be the event that the first card is a king, and B be the event that the second card is a king.

What is the probability of A or B? Clearly

$$\begin{aligned} P[A \textbf{ or } B] &= P[A] + P[B] - P[A \textbf{ and } B] \\ &= 4/52 + 4/52 - (4/52)(4/52) \\ &= .14792. \end{aligned}$$

The first example used the fact that A and B were incompatible. The second used the fact that we could count all the ways in which both A and B could happen. The third used the fact that A and B were independent.