Final Examination

Mth 135 = Sta 104

Friday, 2000 December 15, 2:00 – 5:00 pm

This is a **closed-book** exam. You may use a calculator and a two-sided single sheet of *your own* notes, if you wish, but you may not share materials. A normal distribution table, a page of common pdf/pmf formulas, and a blank sheet are attached. If you don't understand something in one of the questions **please** ask me.

Each problem should take about 10–15 minutes. All problems count equally, even though they are not equally difficult. Point values for problem parts are indicated in parentheses.

You must **show** your **work** to get credit. Unsupported answers aren't acceptable, even if they're correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. Please acknowledge the Duke Honor Code:

I have neither given nor received aid in completing this examination.

Print: ________________ Sign: ________________________________

1.	/10	6.	/10
2.	/10	7.	/10
3.	/10	8.	/10
4.	/10	9.	/10
5.	/10	10.	/10
Total:			/100

Problem 1: Every time Homer opens a bottle of Duff Beer there is a 1/20 probability the bottle will slip out of his hand and smash on the floor and break. Everyone except his wife Marge thinks this is funny (she is the one who cleans up the mess).

a) (5) If he opens 10 Duff Beers, what is the probability that at least one bottle will break? You may assume independence. Give the exact answer, or as a decimal correct to **four places**.

$\mathsf{P}[X \geq 1] =$ ____________

b) (5) Let B_1 be the event, "First Bottle Breaks" and let B_2 be the event, "Second Bottle Breaks". Find (to four decimal places):

$\mathsf{P}[B_1 \cup B_2] =$ ________ $\mathsf{P}[B_1 \cap B_2] =$ ________ $\mathsf{P}[B_1 \mid B_2] =$ ________

Are B_1 and B_2 disjoint? Y N Are B_1 and B_2 independent? Y N

Problem 2: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, or *Uniform* and, whatever the distribution, give its mean μ:

a) The number of times Bart rolls a fair six-sided die before (and not including) the first roll of a six:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

b) The number of red books in Lisa's draw of 6 books (without replacement) from a box containing 10 red books, 10 white books, and 10 black books:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

c) The number of Springfield jurors in a 12-person jury who vote Snake guilty, if 75% of possible jurors believe that he is Guilty and if jurors decide independently:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

d) The number of pot-holes in 3 miles of Springfield's main street, if there are about 5 pot-holes per mile on average and if the numbers of pot-holes in different parts of the road are independent:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

e) The total gross sales of Apu's Kwik-E-Mart in a year, if sales are independent in different weeks and average \$1000/week:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

Problem 3: Nelson breaks Milhouse's 12 cm-long pencil at a random point X cm from the point (hence $Y \equiv 12 - X$ cm from the eraser) with probability density function $f(x) = \frac{1}{12}$, $0 < x < 12$. Be sure to **simplify** your answers below.

a) (3) Find:

$\mathsf{E}[X] =$ __________ $\mathsf{Var}[X] =$ __________

$\mathsf{E}[Y] =$ __________ $\mathsf{Var}[Y] =$ __________

b) (3) Find the covariance and correlation:

$\mathsf{Cov}[X, Y] =$ __________ $\mathsf{Cor}[X, Y] =$ __________

c) (4) Find the probability density function for $R \equiv X/Y$:

$f_R(r) =$ __________

Problem 4: The beer tap in Moe's Tavern can be adjusted to dispense *about* μ ounces (oz) of beer ($0 \leq \mu \leq 20$), but the actual amount dispensed is variable with a standard deviation of $0.2\,oz$. If the amounts dispensed follow a normal probability distribution, how should Moe select μ to ensure that, on average, 95 of every 100 beers poured contain at least a full pint, $16\,oz$?

Problem 5: The purple-haired twins Sherri and Terri are playing a game. Sherri rolls a fair five-sided die; let X be the number shown. Then Terri rolls X fair five-sided dice; let let Y be the sum of the numbers shown on these X dice. Note $1 \leq Y \leq 25$.

a) (4) What are the means of X and Y? (Hint: Use conditioning to find the **marginal** expectation of Y).

$\mathsf{E}[X] =$ ___________ $\mathsf{E}[Y] =$ ___________

b) (2) Are X and Y independent? Circle one and explain: Yes No

c) (4) Find the covariance (Hint: first condition on the value of X):

$\mathsf{Cov}[X, Y] =$ ___________

Problem 6: The Simpsons have had lots of cats, all named Snowball. Let X, Y, Z be the weights of three of them, standardized to have mean zero and variance one. We assume they are independent, normally distributed random variables. Calculate exact answers where possible:

a) (3) $\mathsf{P}[X < Y < Z] =$ ____________ (Hint: This is easy)

b) (4) $\mathsf{P}[X + Y < Z] =$ ____________

c) (3) $\mathsf{E}[(X + Y)(2X - Z)] =$ ____________

Problem 7: The weekly salaries X and Y for Sideshow Bob and Krusty the Klown (in thousands of dollars) are random variables with joint probability density function

$$f(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find:

a) (4) the conditional density function, $f_{X|Y}(x \mid y) =$ ________________

b) (3) the conditional expectation, $\mathsf{E}[X \mid Y = .5] =$ ________________

c) (3) the conditional probability, $\mathsf{P}[X \leq .5 \mid Y = .5] =$ ________________

Problem 8: Itchy and Scratchy have 1,000 fans (go figure). Let X_i be the attention span (in minutes) of the i^{th} fan, and let's assume that the $\{X_i\}$ are independent, all with the exponential distribution with parameter $\lambda = 0.10$ (so X_i has mean $\mu = 10\,\text{min}$ and variance $\sigma^2 = 100\,\text{min}^2$).

a) (5) Find the p.d.f. and the mean for Y, the **minimum** of the 1,000 attention spans (this is the time that the first of Itchy and Scratchy's 1,000 fans will lose interest). (Hint: What must happen for $[Y > y]$ to be true?):

$f_Y(y) =$ ____________ $\mathsf{E}[Y] =$ ____________

b) (5) Find the probability that their **average** fan will have an attention span of at least 10.5 minutes:

$\mathsf{P}[\bar{X}_{1000} > 10.5] =$ ____________

Problem 9: Selma and Patty Bouvier, Marge's twin sisters, smoke *way* too much. Each hour of the workday Selma smokes a Poisson-distributed number S of cigarettes, with mean $\lambda_S = 5$, and Patty smokes (independently) a Poisson-distributed number P of cigarettes with mean $\lambda_P = 4$. Nobody else smokes at the Department of Motor Vehicles where they both work.

a) (8) Chief Wiggum, investigating a theft over lunch hour at the DMV, finds a dirty ashtray that indicates that the thief was a smoker. Assume that it is equally likely to have been Patty's or Selma's ashtray. Chief Wiggum counts the number X of cigarette butts in the ashtray and finds $X = 6$. If these all came from one hour's smoking by one of the twins, what is the probability that the thief was Selma?

$\mathsf{P}[\text{Thief is Selma} \mid X = 6] =$ ________________

b) (2) What is the probability distribution for the length of time, in minutes, until Selma has two more cigarettes? What is the mean?

○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un

$\mu =$

Problem 10: Each time Marge sees her old cellmate, Tattoo Annie, it seems like Annie's hair has a different length X (in inches), with probability density function

$$f(x) = c\,x, \qquad 0 < x < 10.$$

a) (2) Find the value of the constant:

$c =$ ______________

b) (4) Find the indicated expectations:

$\mathsf{E}[X] =$ ______________ $\mathsf{E}[X^2] =$ ______________

c) (4) Find the probability density function for $Y = X^2$:

$f_Y(y) =$ ______________

Done!

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

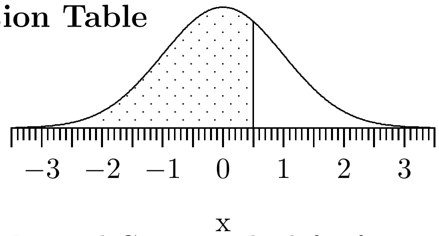

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(x)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(x)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		