Homework 11

Mth 135 = Sta 104

Due: 10:05am Tuesday, November 30, 2004

This is an open-book exam. You may use your books and notes but you may *not* discuss the problems with anyone else, except me. If you don't understand something in one of the questions, or even if you're just stuck, please ask *me*.

You must **show** your **work** to receive credit. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors,
  nor will I accept the actions of those who do.
- I will conduct myself responsibly and honorably in all my
  activities as a Duke student.

Signature: ____________________

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

1. The independent random variables X and Y have uniform distributions on the interval $[0, 1]$. Find:

a. (3) $\mathsf{P}[X > 3Y] =$

b. (3) $\mathsf{E}[X^3 + Y^3] =$

c. (5) $\mathsf{E}[e^{3X}] =$

d. (3) $\mathsf{E}[X \times Y] =$

e. (3) $\mathsf{E}[e^{X-Y}] =$

f. (3) $\mathsf{E}[\sqrt{X/Y}] =$

Hint: *No* 2-dimensional integration is needed, just a little plane geometry.

2. The random variables X and Y have a joint probability density function given by

$$f(x,y) = \begin{cases} 6(y-x) & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a. (8) The marginal probability density functions:

$f_1(x) =$ $f_2(y) =$

b. (4) $\mathsf{P}[Y < 0.50]$

c. (3) $\mathsf{P}[X < Y]$

d. (5) For some positive integers j, k, and n, X and Y have the same distribution as the j'th and k'th smallest $U_{(j)}$ and $U_{(k)}$ of n independent uniformly distributed random variables $U_1, ..., U_n$. What are the values of j, k, and n?

$j =$ ________ $k =$ ________ $n =$ ________

3. The random variables X and Y have independent normal distributions, with means and variances given by:

$$\mathsf{E}[X] = 1 \qquad \mathsf{Var}[X] = 9 \qquad \mathsf{E}[Y] = 2 \qquad \mathsf{Var}[Y] = 16.$$

Without doing any integration, find each of the following:

a. (5) $\mathsf{E}[X^2 + Y^2] =$ ____________________

b. (5) $\mathsf{P}[X < 9 + Y] =$ ____________________

c. (5) $\mathsf{Var}[2\,X - Y] =$ ____________________

d. (5) $\mathsf{E}[X \times Y]$ ____________________

4. The random variable X has an exponential distribution with mean $1/\ln 4$, so it has probability density function and cumulative distribution function

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \lambda e^{-\lambda x} & \text{for } x > 0 \end{cases} \qquad F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0 \end{cases}$$

with $\lambda = \ln 4$, so $\mathsf{P}[X > x] = 4^{-x}$ for all $x \in (0, \infty)$. The random variable $Y \equiv \lfloor X \rfloor$ is X, rounded *down* to the next lower integer... so if X is 3, π, or 3.9999999 then $Y = 3$, but if $X = 4.00000001$ then $Y = 4$.

a. (5) Does Y have a *discrete* distribution or a *continuous* one?
Circle one: Disc Cont and find the p.m.f. or p.d.f.:

$f(y) =$

b. (5) The distribution of Y is one of the ones we've studied— what is it's name, the value of any parameter(s), and its mean? (see pg. 8)

c. (5) Find the indicated probability:

$\mathsf{P}[Y \ge 2] =$

d. (5) Find the conditional probability:

$\mathsf{P}[X \le 2.5 | Y \ge 2] =$

5. The lifetimes (in months) before quixleblats fail are random variables with survival function $\Pr[T > t] = e^{-\sqrt{t}}$, for $t > 0$.

a. (10) Pat has two quixleblats— a brand new one, and a one-month old one. Which has a better chance of lasting for one more month? Why? Show your work.

Check one: ◯ New ◯ Old ◯ Doesn't Matter
Reasoning:

b. (5) Find the probability density function for quixleblat life at *all* $t \in \mathbb{R}$:

$f_T(t) =$ ____________

c. (5) Find the probability density function for $X = \sqrt{T}$ at *all* $x \in \mathbb{R}$:

$f_X(x) =$ ____________

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

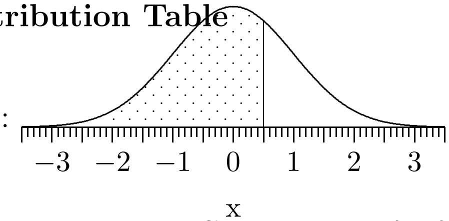

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(x)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(x)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		