Common probability distributions

Distribution	pdf/pmf	Variable	Parameters	Mean	Variance
$\mathsf{Bernoulli}(p)$	$p^x(1-p)^{1-x}$	$x = 0$ or 1	$0 < p < 1$	p	$p(1-p)$
$\mathsf{Binomial}(n,p)$	$\binom{n}{x}p^x(1-p)^{n-x}$	$x = 0, 1, \cdots, n$	$0 < p < 1$	np	$np(1-p)$
$\mathsf{Poisson}(\mu)$	$e^{-\mu}\frac{\mu^x}{x!}$	$x = 0, 1, 2, \cdots$	$\mu > 0$	μ	μ
$\mathsf{Normal}(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$-\infty < x < \infty$	$-\infty < \mu < \infty$, $\sigma > 0$	μ	σ^2
$\mathsf{Exponential}(\lambda)$	$\lambda e^{-\lambda x}$	$x > 0$	$\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$\mathsf{Uniform}(a,b)$	$\frac{1}{b-a}$	$a \le x \le b$	$-\infty < a < b < \infty$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$\mathsf{Beta}(a,b)$	$\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$	$0 < x < 1$	$a > 0, b > 0$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$
$\mathsf{Gamma}(a,b)$	$\frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}$	$x > 0$	$a > 0, b > 0$	$\frac{a}{b}$	$\frac{a}{b^2}$
$\mathsf{Pareto}(a,b)$	$\frac{ab^a}{x^{a+1}}$	$x > b$	$a > 0, b > 0$	$\frac{ab}{a-1}$ if $a > 1$	$\frac{ab^2}{(a-1)^2(a-2)}$ if $a > 2$

Basic probability definitions and facts

Independent normals: If $X \sim \mathsf{Normal}(a, b^2)$, $Y \sim \mathsf{Normal}(c, d^2)$, X & Y are independent, then $X + Y \sim \mathsf{Normal}(a + c, b^2 + d^2)$.

Chi-square: If $Z_1, Z_2, \cdots, Z_k \overset{\text{IID}}{\sim} \mathsf{Normal}(0,1)$ then $V = Z_1^2 + Z_2^2 + \cdots + Z_k^2 \sim \chi^2(k)$.

T: If $W \sim \mathsf{Normal}(0,1)$, $V \sim \chi^2(k)$ and W, V are independent, then $T = \frac{W}{\sqrt{V/k}} \sim t(k)$.

Normal-inverse-chi-square: $(W, V) \sim \mathsf{N}\chi^{-2}(m, k, r, s)$ if and only if $rs/V \sim \chi^2(r)$, $W|(V = v) \sim \mathsf{Normal}(m, v/k)$. Moreover, $T = \frac{W-m}{\sqrt{s/k}} \sim t(r)$.

Normal and $\mathsf{N}\chi^{-2}$: If $(W, V) \sim \mathsf{N}\chi^{-2}(m, k, r, s)$ and $U|(W = w, V = v) \sim \mathsf{Normal}(aw, bv)$ then $\frac{U-am}{\sqrt{s(b+a^2/k)}} \sim t(r)$.

Normal-normal-inverse-chi-square: $(W_1, W_2, V) \sim \mathsf{NN}\chi^{-2}(m_1, k_1, m_2, k_2, r, s)$ if and only if $rs/V \sim \chi^2(r)$, $W_1|(V = v) \sim \mathsf{Normal}(m_1, v/k_1)$, $W_2|(V = v) \sim \mathsf{Normal}(m_2, v/k_2)$ and W_1 & W_2 are conditionally independent given V.

From $\mathsf{NN}\chi^{-2}$ *to* $\mathsf{N}\chi^{-2}$: If $(W_1, W_2, V) \sim \mathsf{NN}\chi^{-2}(m_1, k_1, m_2, k_2, r, s)$ then for any constants a_1, a_2, we must have $(a_1W_1 + a_2W_2, V) \sim \mathsf{N}\chi^{-2}(a_1m_1 + a_2m_2, (a_1^2/k_1 + a_2^2/k_2)^{-1}, r, s)$.

CLT: If $X_1, \cdots, X_n$ are IID each with mean μ and variance σ^2, then $\bar{X} \overset{\text{approx}}{\sim} \mathsf{Normal}(\mu, \frac{\sigma^2}{n})$.

Median: If $X_1, \cdots, X_n \overset{\text{IID}}{\sim} g(x_i|\mu)$ where the pdf $g(x_i|\mu) = g_0(x_i - \mu)$ for some function g_0 symmetric around zero, then $X_{\text{med}} \overset{\text{approx}}{\sim} \mathsf{Normal}(\mu, \frac{1}{4ng_0^2(0)})$.

Normal mean-variance: If $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$ then $\bar{X} \sim \mathsf{Normal}(\mu, \sigma^2/n)$, $(n-1)s_X^2/\sigma^2 \sim \chi^2(n-1)$ and $\bar{X}$ and s_X^2 are independent.

Asymptotic normality of MLE: For a model $X_1, \cdots, X_n \overset{\text{IID}}{\sim} g(x_i|\theta)$, $\theta \in \Theta$ a scalar parameter, if $g(x_i|\theta)$ satisfies certain regularity conditions (including three times differentiability of $\theta \mapsto \log g(x_i|\theta)$, uniqueness of MLE) then for any θ_0 inside of Θ, the random variable $\sqrt{I_X}(\hat{\theta}_{\text{MLE}}(X) - \theta_0) \overset{\text{approx}}{\sim} \mathsf{Normal}(0,1)$ whenever $X_i \overset{\text{IID}}{\sim} g(x_i|\theta_0)$.

Pearson's chi-square: For $X = (X_1, \cdots, X_k) \sim \mathsf{Multinomial}(n, p)$, $p \in \Delta_k$, $Q(X) = \sum_{l=1}^{k} \frac{(X_l - e_l)^2}{e_l} \sim \chi^2(k-1)$ approximately with $e_l = np_l$. If instead $e_l = n\hat{p}_l$, where $\hat{p}$ is an ML estimate of p restricted to a sub-simplex Δ_k^0 determined by r free parameters, then $Q(X) \sim \chi^2(k - 1 - r)$ approximately.

Known triplets of model-prior-posterior

Model	Prior	Posterior
$X \sim \mathsf{Binomial}(n,p)$ $0 \le p \le 1$	$\mathsf{Beta}(a,b)$ $a>0, b>0$	$\mathsf{Beta}(a',b')$ $a' = a+x, b' = b+n-x$
$X = (X_1, \cdots, X_n)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Poisson}(\mu)$ $\mu > 0$	$\mathsf{Gamma}(a,b)$ $a>0, b>0$	$\mathsf{Gamma}(a',b')$ $a' = a+n\bar{x}, b' = b+n$
$X = (X_1, \cdots, X_n)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Exponential}(\lambda)$ $\lambda > 0$	$\mathsf{Gamma}(a,b)$ $a>0, b>0$	$\mathsf{Gamma}(a',b')$ $a' = a+n, b' = b+n\bar{x}$
$X = (X_1, \cdots, X_n)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu,\sigma^2)$	$\mathsf{Normal}(a,b^2)$ $-\infty < a < \infty, b > 0$	$\mathsf{Normal}(a',b'^2)$ $a' = \frac{nb^2\bar{x}+\sigma^2 a}{nb^2+\sigma^2}, b'^2 = \frac{\sigma^2 b^2}{nb^2+\sigma^2}$
$-\infty < \mu < \infty$ σ^2 known	$\xi(\mu) = \text{const}$ (Jeffreys)	$\mathsf{Normal}(\bar{x}, \sigma^2/n)$
$X = (X_1, \cdots, X_n)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu,\sigma^2)$ $-\infty < \mu < \infty$	$\mathsf{N}\chi^{-2}(m,k,r,s)$ $-\infty < m < \infty$ $k>0, r>0, s>0$	$\mathsf{N}\chi^{-2}(m',k',r',s')$ $m' = \frac{km+n\bar{x}}{k+n}, k' = k+n, r' = r+n$ $s' = \frac{rs+\frac{kn}{k+n}(\bar{x}-m)^2+(n-1)s_x^2}{r+n}$
$\sigma^2 > 0$	$\xi(\mu,\sigma^2) = \frac{\text{const}}{\sigma^2}$ (Reference)	$\mathsf{N}\chi^{-2}(\bar{x}, n, n-1, s_x^2)$
$X = (X_1, \cdots, X_n)$ $Y = (Y_1, \cdots, Y_m)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu_1,\sigma^2)$ $Y_j \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu_2,\sigma^2)$	$\mathsf{NN}\chi^{-2}(m_1,k_1,m_2,k_2,r,s)$ $-\infty < m_1, m_2 < \infty$ $k_1 > 0, k_2 > 0,$ $r>0, s>0$	$\mathsf{NN}\chi^{-2}(m_1',k_1',m_2',k_2',r',s')$ $m_1' = \frac{k_1 m_1+n\bar{x}}{k_1+n}, \quad m_2' = \frac{k_2 m_2+m\bar{y}}{k_2+m}$ $k_1' = k_1+n, \, k_2' = k_2+m, r' = r+n$ $s' = \frac{rs+\frac{k_1 n}{k_1+n}(\bar{x}-m_1)^2+\frac{k_2 m}{k_2+m}(\bar{y}-m_2)^2+(n-1)s_x^2+(m-1)s_y^2}{r+n}$
$-\infty < \mu_1, \mu_2 < \infty$ $\sigma^2 > 0$	$\xi(\mu_1,\mu_2,\sigma^2) = \frac{\text{const}}{\sigma^2}$ (Reference)	$\mathsf{NN}\chi^{-2}(\bar{x}, n, \bar{y}, m, n+m-2, \frac{(n-1)s_x^2+(m-1)s_y^2}{n+m-2})$
$X = (X_1, \cdots, X_n)$ $X_i \overset{\text{IID}}{\sim} \mathsf{Uniform}(0,\theta)$ $\theta > 0$	$\mathsf{Pareto}(a,b)$ $a>0, b>0$	$\mathsf{Pareto}(a',b')$ $a' = a+n, \, b' = \max(b, x_{\max})$

$\mathsf{Normal}(0,1)$ **bins with equal probabilities.**

#bins (k)	bin boundaries $(\Phi^{-1}(0/k), \Phi^{-1}(1/k), \cdots, \Phi^{-1}(k/k))$
2	-Inf 0 Inf
3	-Inf -0.43 0.43 Inf
4	-Inf -0.67 0.00 0.67 Inf
5	-Inf -0.84 -0.25 0.25 0.84 Inf
6	-Inf -0.97 -0.43 0.00 0.43 0.97 Inf
7	-Inf -1.07 -0.57 -0.18 0.18 0.57 1.07 Inf
8	-Inf -1.15 -0.67 -0.32 0.00 0.32 0.67 1.15 Inf
9	-Inf -1.22 -0.76 -0.43 -0.14 0.14 0.43 0.76 1.22 Inf
10	-Inf -1.28 -0.84 -0.52 -0.25 0.00 0.25 0.52 0.84 1.28 Inf

ML tests of size α. For one sided tests, $\alpha \le 1/2$. For the fourth model, $r(x,y)$ refers to Welch's approximation. Size calculations for fourth and fifth models are approximate. In the sixth model, $\hat\beta_0 = \bar y - s_{xy}\bar x/s_x^2$ and $\hat\beta_1 = s_{xy}/s_x^2$ denote the least-squares estimates, and $\hat\sigma^2 = \frac{1}{n-2}\sum_{i=1}^n (y_i - \hat\beta_0 - \hat\beta_1 x_i)^2$.

Model	Hypotheses	Size α ML test rejects H_0 if and only if
$X = (X_1, \cdots, X_n)$	$H_0: \mu = \mu_0,\, H_1: \mu \neq \mu_0$	$\mu_0 \notin \bar x \mp z(\alpha)\frac{\sigma}{\sqrt{n}}$
$X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$	$H_0: \mu \le \mu_0, H_1: \mu > \mu_0$	$\mu_0 < \bar x - z(2\alpha)\frac{\sigma}{\sqrt{n}}$
σ known	$H_0: \mu \ge \mu_0, H_1: \mu < \mu_0$	$\mu_0 > \bar x + z(2\alpha)\frac{\sigma}{\sqrt{n}}$
$X = (X_1, \cdots, X_n)$	$H_0: \mu = \mu_0,\, H_1: \mu \neq \mu_0$	$\mu_0 \notin \bar x \mp z_{n-1}(\alpha)\frac{s_x}{\sqrt{n}}$
$X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$	$H_0: \mu \le \mu_0, H_1: \mu > \mu_0$	$\mu_0 < \bar x - z_{n-1}(2\alpha)\frac{s_x}{\sqrt{n}}$
	$H_0: \mu \ge \mu_0, H_1: \mu < \mu_0$	$\mu_0 > \bar x + z_{n-1}(2\alpha)\frac{s_x}{\sqrt{n}}$
$X = (X_1, \cdots, X_n), Y = (Y_1, \cdots, Y_m)$	$H_0: \mu_1 - \mu_2 = \eta_0,\, H_1: \mu_1 - \mu_2 \neq \eta_0$	$\eta_0 \notin (\bar x - \bar y) \mp z_{n+m-2}(\alpha)\sqrt{(\frac{1}{n} + \frac{1}{m})\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}$
$X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$, $Y_j \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu_2, \sigma^2)$	$H_0: \mu_1 - \mu_2 \le \eta_0,\, H_1: \mu_1 - \mu_2 > \eta_0$	$\eta_0 < (\bar x - \bar y) - z_{n+m-2}(2\alpha)\sqrt{(\frac{1}{n} + \frac{1}{m})\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}$
X_i's, Y_j's independent	$H_0: \mu_1 - \mu_2 \ge \eta_0,\, H_1: \mu_1 - \mu_2 < \eta_0$	$\eta_0 > (\bar x - \bar y) + z_{n+m-2}(2\alpha)\sqrt{(\frac{1}{n} + \frac{1}{m})\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}$
$X = (X_1, \cdots, X_n), Y = (Y_1, \cdots, Y_m)$	$H_0: \mu_1 - \mu_2 = \eta_0,\, H_1: \mu_1 - \mu_2 \neq \eta_0$	$\eta_0 \notin (\bar x - \bar y) \mp z_{r(x,y)}(\alpha)\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$
$X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma_1^2)$, $Y_j \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu_2, \sigma_2^2)$	$H_0: \mu_1 - \mu_2 \le \eta_0,\, H_1: \mu_1 - \mu_2 > \eta_0$	$\eta_0 < (\bar x - \bar y) - z_{r(x,y)}(2\alpha)\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$
X_i's, Y_j's independent	$H_0: \mu_1 - \mu_2 \ge \eta_0,\, H_1: \mu_1 - \mu_2 < \eta_0$	$\eta_0 > (\bar x - \bar y) + z_{r(x,y)}(2\alpha)\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}}$
Regular models $X \sim f(x\|\theta)$	$H_0: \theta = \theta_0, H_1: \theta \neq \theta_0$	$\theta_0 \notin \hat\theta_{\text{MLE}}(x) \mp z(\alpha)/\sqrt{I_x}$
(binomial, Poisson, exponential)	$H_0: \theta \le \theta_0, H_1: \theta > \theta_0$	$\theta_0 < \hat\theta_{\text{MLE}}(x) - z(2\alpha)/\sqrt{I_x}$
θ scalar	$H_0: \theta \ge \theta_0, H_1: \theta < \theta_0$	$\theta_0 > \hat\theta_{\text{MLE}}(x) + z(2\alpha)/\sqrt{I_x}$
$Y = (Y_1, \cdots, Y_n)$, $x = (x_1, \cdots, x_n)$	$H_0: a\beta_0 + b\beta_1 = \eta_0,\, H_1: a\beta_0 + b\beta_1 \neq \eta_0$	$\eta_0 \notin (a\hat\beta_0 + b\hat\beta_1) \mp z_{n-2}(\alpha)\hat\sigma\sqrt{\frac{a^2}{n} + \frac{(a\bar x - b)^2}{(n-1)s_x^2}}$
$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $\epsilon_i \overset{\text{IID}}{\sim} \mathsf{Normal}(0, \sigma^2)$	$H_0: a\beta_0 + b\beta_1 \le \eta_0,\, H_1: a\beta_0 + b\beta_1 > \eta_0$	$\eta_0 < (a\hat\beta_0 + b\hat\beta_1) \mp z_{n-2}(2\alpha)\hat\sigma\sqrt{\frac{a^2}{n} + \frac{(a\bar x - b)^2}{(n-1)s_x^2}}$
	$H_0: a\beta_0 + b\beta_1 \ge \eta_0,\, H_1: a\beta_0 + b\beta_1 < \eta_0$	$\eta_0 > (a\hat\beta_0 + b\hat\beta_1) \mp z_{n-2}(2\alpha)\hat\sigma\sqrt{\frac{a^2}{n} + \frac{(a\bar x - b)^2}{(n-1)s_x^2}}$

Table of $z_k(\alpha) = \Phi_k^{-1}(1-\alpha/2)$. Φ_k is $t(k)$ CDF. $z_\infty(\alpha) = z(\alpha)$ is for $\mathsf{Normal}(0,1)$.

	α						
k	.1	.05	.01	.001	.0001	.00001	.000001
9	1.83	2.26	3.25	4.78	6.59	8.83	11.64
10	1.81	2.23	3.17	4.59	6.21	8.15	10.52
11	1.8	2.2	3.11	4.44	5.92	7.65	9.7
12	1.78	2.18	3.05	4.32	5.69	7.26	9.08
13	1.77	2.16	3.01	4.22	5.51	6.95	8.6
14	1.76	2.14	2.98	4.14	5.36	6.71	8.22
15	1.75	2.13	2.95	4.07	5.24	6.5	7.9
16	1.75	2.12	2.92	4.01	5.13	6.33	7.64
17	1.74	2.11	2.9	3.97	5.04	6.18	7.42
18	1.73	2.1	2.88	3.92	4.97	6.06	7.23
19	1.73	2.09	2.86	3.88	4.9	5.95	7.07
20	1.72	2.09	2.85	3.85	4.84	5.85	6.93
21	1.72	2.08	2.83	3.82	4.78	5.77	6.8
22	1.72	2.07	2.82	3.79	4.74	5.69	6.69
23	1.71	2.07	2.81	3.77	4.69	5.63	6.59
24	1.71	2.06	2.8	3.75	4.65	5.57	6.5
∞	1.64	1.96	2.58	3.29	3.89	4.42	4.89

Table of $\Phi_k(c)$. Φ_k is $t(k)$ CDF. $\Phi_\infty = \Phi$, the $\mathsf{Normal}(0,1)$ CDF.

	c										
k	1.5	1.7	1.9	2.1	2.3	2.5	2.7	2.9	3.1	3.3	3.5
9	0.916	0.938	0.955	0.967	0.977	0.983	0.988	0.991	0.994	0.995	0.997
10	0.918	0.94	0.957	0.969	0.978	0.984	0.989	0.992	0.994	0.996	0.997
11	0.919	0.941	0.958	0.97	0.979	0.985	0.99	0.993	0.995	0.996	0.998
12	0.92	0.943	0.959	0.971	0.98	0.986	0.99	0.993	0.995	0.997	0.998
13	0.921	0.944	0.96	0.972	0.981	0.987	0.991	0.994	0.996	0.997	0.998
14	0.922	0.944	0.961	0.973	0.981	0.987	0.991	0.994	0.996	0.997	0.998
15	0.923	0.945	0.962	0.973	0.982	0.988	0.992	0.995	0.996	0.998	0.998
16	0.923	0.946	0.962	0.974	0.982	0.988	0.992	0.995	0.997	0.998	0.999
17	0.924	0.946	0.963	0.975	0.983	0.989	0.992	0.995	0.997	0.998	0.999
18	0.925	0.947	0.963	0.975	0.983	0.989	0.993	0.995	0.997	0.998	0.999
19	0.925	0.947	0.964	0.975	0.984	0.989	0.993	0.995	0.997	0.998	0.999
20	0.925	0.948	0.964	0.976	0.984	0.989	0.993	0.996	0.997	0.998	0.999
21	0.926	0.948	0.964	0.976	0.984	0.99	0.993	0.996	0.997	0.998	0.999
22	0.926	0.948	0.965	0.976	0.984	0.99	0.993	0.996	0.997	0.998	0.999
23	0.926	0.949	0.965	0.977	0.985	0.99	0.994	0.996	0.997	0.998	0.999
24	0.927	0.949	0.965	0.977	0.985	0.99	0.994	0.996	0.998	0.998	0.999
∞	0.933	0.955	0.971	0.982	0.989	0.994	0.997	0.998	0.999	1	1

Table of $F_k^{-1}(1-\alpha)$. F_k is $\chi^2(k)$ CDF.

	k								
α	1	2	3	4	5	6	7	8	9
0.3	1.07	2.41	3.66	4.88	6.06	7.23	8.38	9.52	10.66
0.2	1.64	3.22	4.64	5.99	7.29	8.56	9.8	11.03	12.24
0.1	2.71	4.61	6.25	7.78	9.24	10.64	12.02	13.36	14.68
0.05	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92
0.01	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67
0.001	10.83	13.82	16.27	18.47	20.52	22.46	24.32	26.12	27.88
0.0001	15.14	18.42	21.11	23.51	25.74	27.86	29.88	31.83	33.72