STA 114: Statistics

HW 6

Due Wed Oct 19 2011

1. Lactic acid concentration measurements X_i of samples from a cheese slab are modeled as $X_i \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$, $(\mu, \sigma^2) \in (-\infty, \infty) \times (0, \infty)$ with prior $\xi(\mu, \sigma^2) = \mathsf{N}\chi^{-2}(0, 1/2, 1/2, 1)$.
   (a) Write down (give name and parameter values of) the posterior distribution on (μ, σ^2) based on the following 10 measurements

   `0.86 1.53 1.57 1.81 0.99 1.09 1.29 1.78 1.29 1.58`

   (b) Take the posterior distribution from part (a) as a new prior distribution on (μ, σ^2). For this new prior, write down the posterior distribution on (μ, σ^2) based on 20 additional observations:

   `1.68 1.90 1.06 1.30 1.52 1.74 1.16 1.49 1.63 1.99`
   `1.15 1.33 1.44 2.01 1.31 1.46 1.72 1.25 1.08 1.25`

   (c) Now consider the original prior $\mathsf{N}\chi^{-2}(0, 1/2, 1/2, 1)$ and write down the posterior distribution of (μ, σ^2) based on all 30 observations listed above.
   (d) Are your answers from part (b) and (c) the same? If "yes", explain whether this is a coincidence. If "no", explain whether this should always be the case.

2. For the the model and the observed data of problem 1 above (with all 30 observations), give central 95% prior and posterior credible intervals for σ^2.

3. For each of the following models, find the Jeffreys prior for the model parameter and identify the corresponding posterior distribution.
   (a) $X \sim \mathsf{Binomial}(n, p)$ with parameter $p \in [0, 1]$.
   (b) $X = (X_1, \cdots, X_n)$, $X_i \overset{\text{IID}}{\sim} \mathsf{Exponential}(\lambda)$ with parameter $\lambda \in (0, \infty)$.
   (c) $X = (X_1, \cdots, X_n)$, $X_i \overset{\text{IID}}{\sim} \mathsf{Poisson}(\mu)$ with parameter $\mu \in (0, \infty)$.

4. Consider a two parameter model $X \sim f(x|\theta_1, \theta_2)$, $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$, where both Θ_1 and Θ_2 are discrete sets. Consider a prior pmf $\xi(\theta_1, \theta_2) = \xi_1(\theta_1)\xi_2(\theta_2)$ on (θ_1, θ_2) and the corresponding posterior pmf $\xi(\theta_1, \theta_2|x)$ given an observation $X = x$. Show that the corresponding posterior pmf $\xi_2(\theta_2|x)$ of θ_2 equals the posterior pmf derived from the model $X \sim g(x|\theta_2)$, $\theta_2 \sim \xi_2(\theta_2)$ and observation $X = x$, where $g(x|\theta_2) = \sum_{\theta_1 \in \Theta_1} f(x|\theta_1, \theta_2)\xi(\theta_1)$.

5. Consider the model $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \mathsf{Normal}(\mu, \sigma^2)$, $(\mu, \sigma^2) \in (-\infty, \infty) \times (0, \infty)$ with a product prior pdf $\xi(\mu, \sigma^2) = \xi_1(\mu)\xi_2(\sigma^2)$ where $\xi_1(\mu) = \mathsf{Normal}(a, b^2)$ and[1] $\xi_2(\sigma^2) = \chi^{-2}(r, s)$ for some constants $-\infty < a < \infty$, $b > 0$, $r > 0$ and $s > 0$. Let $\xi(\mu, \sigma^2|x)$ denote the posterior pdf given observations $x = (x_1, \cdots, x_n)$.

[1] Definition: A random variable $V \sim \chi^{-2}(r, s)$ if $\frac{rs}{V} \sim \chi^2(r)$. The pdf of the $\chi^{-2}(r, s)$ distribution is $f(v) = \text{const.} \times v^{-(r+2)/2} \exp(-\frac{rs}{2v})$ for $v > 0$ and $f(v) = 0$ oherwise. The constant equals $(rs/2)^{r/2}/\Gamma(r/2)$.

(a) Identify the conditional posterior pdf $\xi_1(\mu|x, \sigma^2)$ of μ for a given σ^2.

(b) Identify the conditional posterior pdf $\xi_2(\sigma^2|x, \mu)$ of σ^2 for a given μ.