## STA 611: Introduction to Statistical Methods – Fall 2012

# Homework 9 – due November 15, 2012

These exercises are meant to be representative of the material in Chapters 8.6-8.7 and the first part of 9.1 in DeGroot and Schervish.

1. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, 1/\tau)$, where $\tau$ is the precision parameter (reciprocal of the variance). You want to assign a Normal-Gamma prior distribution to $(\mu, \tau)$ but you need to assign values to the hyperparameters $\mu_0$, $\lambda_0$, $\alpha_0$ and $\beta_0$ that reflect your prior knowledge. It can be hard to describe prior knowledge in therms of these parameters, but you might have a good idea about the *marginal* means and standard deviations of $\mu$ and $\tau$. Suppose your prior values of the marginal means and variances are the following:

$$E(\mu) = 10, \quad \text{Var}(\mu) = 2^2, \quad E(\tau) = 1, \quad \text{and} \quad \text{Var}(\tau) = 0.5^2 \ .$$

   Give the hyperparameters $\mu_0$, $\lambda_0$, $\alpha_0$ and $\beta_0$ of the corresponding Normal-Gamma prior distribution.

2. Consider again the measurements you saw in homework 8, problem 6:

$$0.95, 0.85, 0.92, 0.95, 0.93, 0.86, 1.00, 0.92, 0.85, 0.81$$
$$0.78, 0.93, 0.93, 1.05, 0.93, 1.06, 1.06, 0.96, 0.81, 0.96$$

   Again, assume that these data are observations of independent $N(\mu, \sigma^2)$ random variables where $\mu$ and $\sigma^2$ are unknown. This time, use Bayesian methods to analyse the data and in stead of the variance parameter use the precision: $\tau = 1/\sigma^2$.

   (a) Assume the Normal-Gamma prior distribution for $(\mu, \tau)$ with parameters $\mu_0 = 1$, $\lambda_0 = 1.5$, $\alpha_0 = 9$ and $\beta_0 = 3$.

      i. Find the 95% prior credible interval for $\mu$

      ii. State the posterior distribution of $(\mu, \tau)$ and give the posterior hyperparameters.

      iii. Find the 95% posterior credible interval for $\mu$

      iv. Compare the prior and posterior credible intervals

   (b) This time, assume the improper prior $p(\mu, \tau) = \frac{1}{\tau}$ for $-\infty < \mu < \infty$ and $\tau > 0$.

      i. State the posterior distribution of $(\mu, \tau)$ and give the posterior hyperparameters.

      ii. Find the 95% posterior credible interval for $\mu$

3. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, 1/\tau)$. Let $\mu$ and $\tau$ be a priori independent, i.e. $p(\mu, \tau) = p(\mu)p(\tau)$ for all $\mu$ and $\tau$ where

   - $p(\mu) = 1$ for $-\infty < \mu < \infty$ (improper prior for $\mu$) and
   - $p(\tau)$ is the pdf of the Gamma$(\alpha_0, \beta_0)$ distribution

   Show that the posterior distribution of $(\mu, \tau)$ is a Normal-Gamma distribution and give the posterior hyperparameters.

4. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, 1/\tau)$. Let $\mu$ and $\tau$ be a priori independent, i.e. $p(\mu, \tau) = p(\mu)p(\tau)$ for all $\mu$ and $\tau$ where

$$\mu \sim N(\mu_0, 1/\nu_0) \qquad \text{and} \qquad \tau \sim \text{Gamma}(\alpha_0, \beta_0) \ .$$

   Show that the posterior distribution of $(\mu, \tau)$ is NOT a Normal-Gamma distribution.

5. Let $X_1, \ldots, X_{10}$ be a random sample from an unknown distribution and let the following be the observed values:

$$4.18, 4.53, 7.22, 6.77, 5.97, 6.83, 5.53, 6.36, 4.48, 5.63$$

   Give unbiased estimators of $E(X_1)$ and $\text{Var}(X_1)$ and calculate the corresponding observed estimates.

6. Let $X_1, \ldots, X_n$ i.i.d. Uniform$[0, \theta]$. The Maximum Likelihood Estimator (MLE) of $\theta$ is $\hat{\theta} = \max(X_1, \ldots, X_n)$.

   (a) Show that the MLE is a biased estimator of $\theta$ and find the bias.

   (b) Find an unbiased estimator of $\theta$

   (c) Compare the two estimators in terms of their Mean Squared Error (MSE)

7. In each of the following examples assume that you have a random sample from $N(\mu, \sigma^2)$ where $\sigma^2$ is known. In each case determine the null and alternative hypotheses and justify your answer by describing the type I and type II errors.

   (a) In an American car factory, car panels are sprayed by a machine. The paint thickness needs to be 0.225mm on average to properly protect the panels. You are a quality control inspector for the company and you select a random sample of panels and measure the paint thickness at specified points on the panels.

   (b) Customers at a fast food restaurant in a big city spend on average 9 minutes in the dining hall eating their food. The manager believes that playing heavy metal music in the dining hall will make the customers eat faster and therefore make room for more customers. The staff don't like heavy metal (they exclusively listen to classical music) but they agree to do an experiment. For one day they blasted the best of the best in heavy metal and measured how long each customer spent in the dining hall.

8. Let $X_1, \ldots, X_n$ be random sample from $N(\theta, \sigma^2)$ where $\sigma^2 = 9$ and say you want to test the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta \neq \theta_0$$

using the following test: Reject $H_0$ if and only if $T = |\overline{X}_n - \theta_0| \geq c$. For $n = 100$ find the value of $c$ so that the test is a size 0.01 test.

9. Let $X_1, \ldots, X_n$ be a random sample from the Poisson($\lambda$) distribution. Say we want to test the hypotheses

$$H_0 : \lambda \geq 1 \quad \text{and} \quad H_1 : \lambda < 1$$

using the following test: Reject $H_0$ if and only if $Y = \sum_{i=1}^n X_i \leq c$ for some constant $c$.

   (a) Determine the power function of the test

   (b) For $n = 10$ find the largest $c$ so that the test is a level 0.05 test. Find the size of that test.