Chapter 3 sections

- 3.1 Random Variables and Discrete Distributions
- 3.2 Continuous Distributions
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- 3.4 Bivariate Distributions
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- Just skim: 3.7 Multivariate Distributions (generalization of bivariate - random vectors)
- 3.8 Functions of a Random Variable
- 3.9 Functions of Two or More Random Variables
- SKIP: 3.10 Markov Chains

Bivariate discrete distributions

Def: Discrete joint distribution / joint pf

Let X and Y be random variables. If there are at most countable possible outcomes (x, y) for the pair (X, Y), we say that X and Y have a *discrete joint distribution*.
The *joint probability function (joint pf)* is

$$f(x, y) = P(X = x \text{ and } Y = y) =: P(X = x, Y = y) \quad \forall (x, y) \in \mathbb{R}^2$$

As for univariate case we have $f(x, y) \geq 0$ and

$$\sum_{\text{All } (x,y)\in\mathbb{R}^2} f(x, y) = 1$$

and

$$P((X, Y) \in C) = \sum_{(x,y)\in C} f(x, y)$$

Example - Three coin tosses

A fair coin is tossed three times. Let

- $X =$ number of heads on the first toss
- $Y =$ total number of heads

The pf $f(x, y)$ can be given in a table:

x \ y	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

Can easily see that $\sum_{(x,y)} f(x, y) = 1$

Bivariate continuous distributions

Def: Continuous joint distribution / joint pdf

Two random variables X and Y have a *continuous joint distribution* if there exists a non-negative function f such that for every $C \subset \mathbb{R}^2$

$$P((X, Y) \in C) = \int_C \int f(x, y) dxdy$$

The function f is called the *joint probability density function (joint pdf)*.

A joint pdf must satisfy:

$$f(x, y) \geq 0 \quad -\infty < x < \infty, \ -\infty < y < \infty$$

$$\text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$$

- Mixed discrete and continuous variables: Use integrals for continuous dimension, and sums for discrete dimension.

Example

Verify that

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a joint pdf

Bivariate cumulative distribution function

Def: Joint cumulative distribution function

The *joint cumulative distribution function (joint cdf)* of two random variables X and Y is

$$F(x,y) = P(X \leq x, X \leq y) \quad \forall (x,y) \in \mathbb{R}^2$$

Relationship between joint cdf's and joint pdf's:

- Continuous:

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(r,s) drds$$

and

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x}$$

- Discrete:

$$F(x,y) = \sum_{r \leq x} \sum_{s \leq y} f(r,s)$$

Example

Find the joint cdf for the following joint pdf

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal distributions - discrete random variables

Theorem

Let (X, Y) be a discrete random vector with joint pf $f_{X,Y}(x, y)$, then the *marginal pfs* of X and Y are given by

$$f_X(x) = P(X = x) = \sum_{y \in \mathbb{R}} f(x, y)$$

$$\text{and } f_Y(y) = P(Y = y) = \sum_{x \in \mathbb{R}} f(x, y)$$

Example: Find the marginal distributions for the coin toss example

Marginal distributions - continuous random variables

Theorem

Let (X, Y) be a continuous random vector with joint pdf $f_{X,Y}(x, y)$, then the *marginal pdfs* of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy \quad \text{for } -\infty < x < \infty$$

$$\text{and } f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx \quad \text{for } -\infty < y < \infty$$

Example: Find the marginal distributions for

$$f(x, y) = 8xy \quad \text{for } 0 < y < x < 1$$

Independence

Independence for random variables is defined in the same way as for events

Def: Independent random variables

Two random variables are *independent* if for every two sets A and B in $\mathbb{R}$ the events $\{s : X(s) \in A\}$ and $\{s : Y(s) \in B\}$ are independent events

Theorem

Random variables X and Y are independent if and only if

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

Independence

The following holds for both discrete and continuous random variables:

Theorem

Two random variables X and Y with joint pf/pdf $f(x, y)$ and marginal pf's/pdf's $f_X(x)$ and $f_Y(y)$ are independent if and only if

$$f(x, y) = f_X(x) f_Y(y)$$

for ALL $(x, y) \in \mathbb{R}^2$

Examples: Are the following random variables independent?

1. X and Y in the tossing coin example
2. X and Y with joint pdf $f(x, y) = 6xy^2$ for $0 < y < 1$ and $0 < x < 1$
3. X and Y with joint pdf $f(x, y) = 8xy$ for $0 < y < x < 1$

Independence

A helpful theorem

Theorem

Let X and Y be random variables with joint pf/pdf $f(x, y)$ and support that is a rectangle R in $\mathbb{R}^2$ (possibly unbounded).

Then X and Y are independent if and only if f can be written as

$$f(x, y) = h_1(x)h_2(y)$$

for all $(x, y) \in R$

Conditional distributions

Def: Conditional distribution

Let X and Y be random variables with joint pf/pdf $f(x, y)$. Let $f_Y(y)$ be the marginal pf/pdf of Y and let y be a value such that $f_Y(y) > 0$. Then the *conditional pf/pdf of X given that $Y = y$* is defined as

$$f(x|y) = \frac{f(x, y)}{f_Y(y)}$$

- Note that in the continuous case we are conditioning on something that has probability 0. We need to show that the continuous case of $f(x|y)$ is indeed a pdf

Examples: Find the conditional pf/pdf:

- $X|Y = 2$ from the tossing coin example
- $X|Y = y$ where the joint pdf is $f(x, y) = 8xy$ for $0 < y < x < 1$

Independence and conditional distributions

Theorem

Random variables X and Y are independent if and only if

$$f(x|y) = f_X(x)$$

- We have the law of total probability for random variables (Theorem 3.6.3 in the book)
- We also have Bayes' theorem for random variables (Theorem 3.6.4 in the book)

Multivariate Distributions - extension of bivariate

- Random vector: $\mathbf{X} = (X_1, X_2, \dots, X_n)$
- Joint cdf:

$$F(\mathbf{x}) = F(x_1, x_2, \dots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n)$$

- Discrete joint pf:

$$\begin{aligned} f(\mathbf{x}) &= f(x_1, x_2, \dots, x_n) \\ &= P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = P(\mathbf{X} = \mathbf{x}) \end{aligned}$$

- Continuous joint pf:

$$f(\mathbf{x}) = f(x_1, x_2, \dots, x_n) = \frac{\partial^n F(x_1, x_2, \dots, x_n)}{\partial x_1 \cdots \partial x_n}$$

$$P(\mathbf{X} \in C) = \int \underset{C}{\cdots} \int f(x_1, x_2, \dots, x_n) dx_1 \cdots dx_n$$

Multivariate Distributions - extension of bivariate

- Marginal pdf - integrate out all the others, e.g:

$$f_1(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_2 \cdots dx_n$$

- $X_1, \ldots, X_n$ are *independent* if for every set $A_1, \ldots, A_n$ in $\mathbb{R}$

$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \times \cdots \times P(X_n \in A_n)$$

- $X_1, \ldots, X_n$ are independent if and only if

$$F(x_1, x_2, \ldots, x_n) = F_1(x_1) \times F_2(x_2) \times \cdots \times F_n(x_n)$$

- $X_1, \ldots, X_n$ are independent if and only if

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n)$$

- *Conditional pdfs*

$$\begin{aligned} f(\mathbf{x}|\mathbf{y}) &= f(x_1, x_2, \ldots, x_n | y_1, y_2, \ldots, y_k) \\ &= \frac{f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k)}{f_{\mathbf{Y}}(y_1, y_2, \ldots, y_k)} = \frac{f(\mathbf{x}, \mathbf{y})}{f_{\mathbf{Y}}(\mathbf{y})} \end{aligned}$$