Chapter 8: Sampling distributions of estimators

Sections

- 8.1 Sampling distribution of a statistic
- 8.2 The Chi-square distributions
- 8.3 Joint Distribution of the sample mean and sample variance
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- 8.4 The *t* distributions
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- 8.5 Confidence intervals
- 8.6 Bayesian Analysis of Samples from a Normal Distribution
- 8.7 Unbiased Estimators
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Unbiased Estimators

We are back in the Frequentist realm!

- Say we are interested in estimating $g(\theta)$
- It is desirable that the estimator we use, $\delta(\mathbf{X})$, will be close to $g(\theta)$ with high probability
- We want the distribution of $\delta(\mathbf{X})$ to be concentrated around $g(\theta)$
- Example: Consider $\delta(\mathbf{X}) = \overline{X}_n$ as an estimator of θ in $N(\theta, \sigma^2)$. Since $\overline{X}_n \sim N(\theta, \sigma^2/n)$ this estimator will be concentrated around θ, no matter what the value of θ is

Unbiased Estimators and Bias

Def: Unbiased Estimator / Bias

An estimator $\delta(\mathbf{X})$ is an *unbiased estimator of $g(\theta)$* if

$$E(\delta(\mathbf{X})) = g(\theta) \qquad \forall\theta \, .$$

Otherwise it is called a *biased estimator*. The *bias* is defined as

$$E(\delta(\mathbf{X})) - g(\theta)$$

Examples:

- $X_1, \dots, X_n$ i.i.d. $N(\mu, \sigma^2)$. $\overline{X}_n$ is an unbiased estimator of μ since $E(\overline{X}_n) = \mu$ for all μ
- Let $X_1, \dots, X_n$ be i.i.d. $\mathrm{Expo}(\theta)$.
  1. Show that the MLE of θ is a biased estimator of θ and find the bias
  2. Modify the MLE so that you have an unbiased estimator of θ

Mean Square Error (MSE)

- Is unbiased good enough?
- Useless if the estimator has high variance
- Look for unbiased estimators with lowest variance
- Recall: Mean squared error: $E\left((\delta(\mathbf{X}) - g(\theta))^2\right)$
- Want estimators with small MSE.

Corollary 8.7.1

Let $\delta(\mathbf{X})$ be an estimator with finite variance. Then

$$\text{MSE}(\delta(\mathbf{X})) = \mathrm{Var}(\delta(\mathbf{X})) + \text{bias}(\delta(\mathbf{X}))^2$$

$\Rightarrow$ the MSE of an unbiased estimator is equal to its variance.

- Searching for unbiased estimator with small variance is equivalent to searching for unbiased estimators with small MSE.

Example

Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{Expo}(\theta)$.

- Consider three estimators of θ
  - $\delta_1 = n/T$ (the MLE of θ)
  - $\delta_2 = (n-1)/T$ (unbiased)
  - $\delta_3 = (n-2)/T$
- Find the MSE of each estimator.
- Which estimator has smaller MSE?
- Which estimator do you prefer?

Unbiased estimators of mean and variance

From any distribution

Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta)$. The mean and variance of the distribution (if exist) are functions of θ.

Unbiased estimation of the mean

- Example 8.7.4: If the mean and variance are finite then $\overline{X}_n$ is an unbiased estimator of the mean $E(X_1)$ and has MSE $= \mathrm{Var}(X_1)/n$. (We knew this already)

Unbiased estimation of the variance

- Theorem 8.7.1: If variance is finite then $\hat{\sigma}_1^2$ is an unbiased estimator of $\mathrm{Var}(X)$ where

$$\hat{\sigma}_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

- Note: This means that the MLE of σ^2 in $N(\mu, \sigma^2)$ is a biased estimator

Why unbiased?

- Sounds good - who wants to be “biased”?
- However, the variance or MSE are better evaluators of quality of estimators
- In many cases there exist biased estimators with smaller MSE (see the exponential example)
- There may not exists an unbiased estimator

END OF CHAPTER 8