Unit 7: Multiple Linear Regression
Lecture 3: Confidence and prediction intervals & Transformations

Statistics 101

Mine Çetinkaya-Rundel

November 26, 2013

Announcements

- PA7 – Last PA!
  - opens at 5pm today, due by midnight on Monday (Dec 2)
- Poster sessions: Dec 2 @ the Link
  - Section 1 (10:05 - 11:20, George) - Link Classroom 4
  - Section 2 (11:45 - 1:00, George) - Link Classroom 5
  - Section 3 (1:25 - 2:40, Daiana & Christine) - Link Classroom 5
  - Section 4 (3:05 - 4:20, Anthony) - Link Classroom 4
  - Section 5 (4:40 - 5:55, Daiana) - Link Classroom 5
- Questions over the break:
  - Piazza
  - Sunday, Dec 1 - George OH - 8-9pm
- Papers: due Thursday, Dec 5
  - hard copy in class
  - markdown file on Sakai
  - only one submission per team on Sakai

Uncertainty of predictions

- Regression models are useful for making predictions for new observations not include in the original dataset.
- If the model is good, the predictions should be close to the true value of the response variable for this observation, however it may not be exact, i.e. $\hat{y}$ might be different than y.
- With any prediction we can (and should) also report a measure of uncertainty of the prediction:
  - Use a confidence interval for the uncertainty around the expected value of predictions (average of a group of predictions) – e.g. predict the average final exam score of a group of students who scored the same on the midterm.
  - Use a prediction interval for the uncertainty around a single prediction – e.g. predict the final exam score of one student with a given midterm score.

Confidence intervals for average values

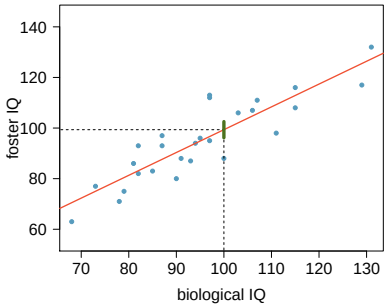
A confidence interval for the average (expected) value of y, $E(y)$, for a given $x^\star$, is

$$\hat{y} \pm t^\star_{n-2} s \sqrt{\frac{1}{n} + \frac{(x^\star - \bar{x})^2}{(n-1)s_x^2}}$$

where s is the standard deviation of the residuals, calculated as $\sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n-2}}$.

Calculate a 95% confidence interval for the average IQ score of foster twins whose biological twins have IQ scores of 100 points. Note that the average IQ score of 27 biological twins in the sample is 95.3 points, with a standard deviation is 15.74 points.

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   9.20760    9.29990   0.990    0.332
bioIQ         0.90144    0.09633   9.358  1.2e-09

Residual standard error: 7.729 on 25 degrees of freedom
```



$$\begin{aligned} \hat{y} &= 9.2076 + 0.90144 \times 100 \approx 99.35 \\ df &= n - 2 \qquad t^\star = 2.06 \\ ME &= 2.06 \times 7.729 \times \sqrt{\frac{1}{27} + \frac{(100 - 95.3)^2}{26 \times 15.74^2}} \\ &\approx 3.2 \\ CI &= 99.35 \pm 3.2 \\ &= (96.15, 102.55) \end{aligned}$$

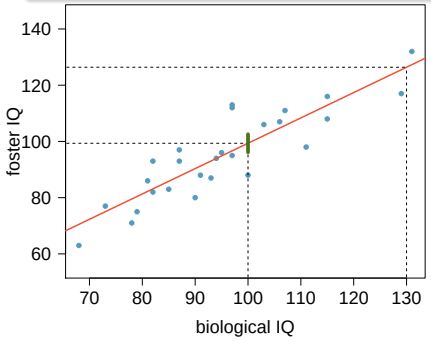
Confidence interval for a prediction – in R

```
# load data
install.packages("faraway") # dataset can be found in this package
library(faraway)
data(twins)

# fit model
m = lm(Foster ~ Biological, data = twins)

# create a new data frame for the new observation
newdata = data.frame(Biological = 100)

# calculate a prediction
# and a confidence interval for the prediction
predict(m , newdata, interval = "confidence")
```

```
     fit      lwr      upr
99.3512 96.14866 102.5537
```


Clicker question

How would you expect the width of the 95% confidence interval for the average IQ score of foster twins whose biological twins have IQ scores of 130 points ($x^\star = 130$) to compare to the previous confidence interval (where $x^\star = 100$)?



(a) wider
(b) narrower
(c) same width
(d) cannot tell

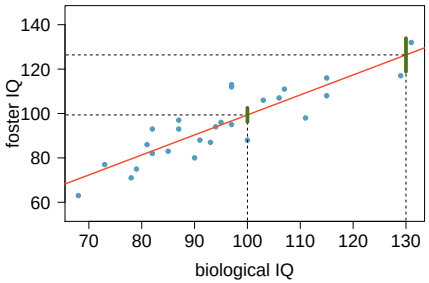
How do the confidence intervals where $x^\star = 100$ and $x^\star = 130$ compare in terms of their widths?

$x^\star = 100$

$$ME_{100} = 2.06 \times 7.729 \times \sqrt{\frac{1}{27} + \frac{(100 - 95.3)^2}{26 \times 15.74^2}} = 3.2$$

$x^\star = 130$

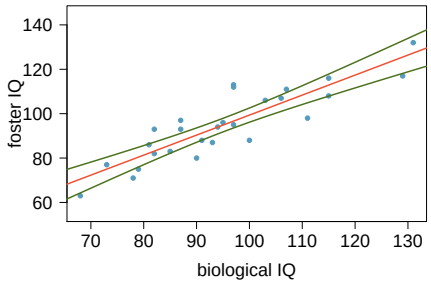
$$ME_{130} = 2.06 \times 7.729 \times \sqrt{\frac{1}{27} + \frac{(130 - 95.3)^2}{26 \times 15.74^2}} = 7.53$$



Recap

The width of the confidence interval for $E(y)$ increases as $x^\star$ moves away from the center.

- Conceptually: We are much more certain of our predictions at the center of the data than at the edges (and our level of certainty decreases even further when predicting outside the range of the data – extrapolation).
- Mathematically: As $(x^\star - \bar{x})^2$ term increases, the margin of error of the confidence interval increases as well.



Clicker question

Earlier we learned how to calculate a confidence interval for average y, $E(y)$, for a given $x^\star$.

Suppose we're not interested in the average, but instead we want to predict a future value of y for a given $x^\star$.

Would you expect there to be more uncertainty around an average or a specific predicted value?

(a) more uncertainty around an average
(b) more uncertainty around a specific predicted value
(c) equal uncertainty around both values
(d) cannot tell

Prediction intervals for specific predicted values

A *prediction interval* for y for a given $x^\star$ is

$$\hat{y} \pm t^\star_{n-2} s \sqrt{1 + \frac{1}{n} + \frac{(x^\star - \bar{x})^2}{(n-1)s_x^2}}$$

where s is the standard deviation of the residuals.

- The formula is very similar, except the variability is higher since there is an added 1 in the formula.
- Prediction level: If we repeat the study of obtaining a regression data set many times, each time forming a XX% prediction interval at $x^\star$, and wait to see what the future value of y is at $x^\star$, then roughly XX% of the prediction intervals will contain the corresponding actual value of y.

Application exercise: Prediction interval

Calculate a 95% prediction interval for the IQ score of a foster twin whose biological twin has an IQ score of 100 points. Note that the average IQ score of 27 biological twins in the sample is 95.3 points, with a standard deviation is 15.74 points.

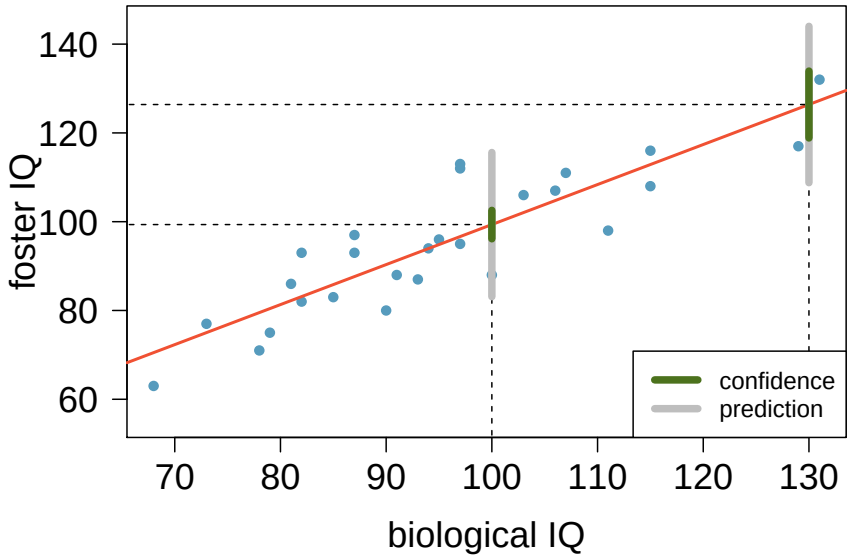
```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   9.20760    9.29990   0.990    0.332
bioIQ         0.90144    0.09633   9.358  1.2e-09

Residual standard error: 7.729 on 25 degrees of freedom
```

Confidence interval for a prediction – in R

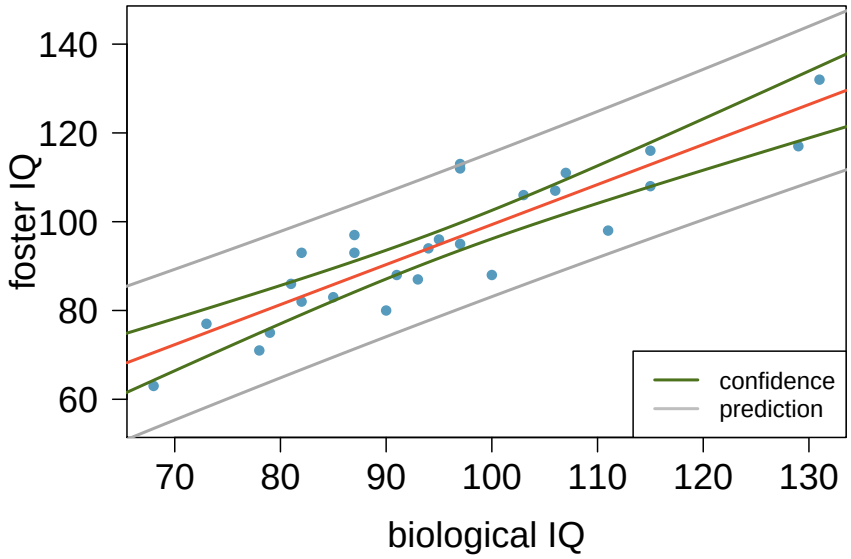
```
# calculate a prediction
# and a confidence interval for the prediction
predict(m , newdata, interval = "prediction")
```

```
     fit      lwr      upr
 99.3512 83.11356 115.5888
```


CI for $E(y)$ vs. PI for y (1)



CI for $E(y)$ vs. PI for y (2)



CI for $E(y)$ vs. PI for y - differences

- A prediction interval is similar in spirit to a confidence interval, except that
  - the prediction interval is designed to cover a “moving target”, the random future value of y, while
  - the confidence interval is designed to cover the “fixed target”, the average (expected) value of y, $E(y)$,

  for a given $x^\star$.
- Although both are centered at $\hat{y}$, the prediction interval is wider than the confidence interval, for a given $x^\star$ and confidence level. This makes sense, since
  - the prediction interval must take account of the tendency of y to fluctuate from its mean value, while
  - the confidence interval simply needs to account for the uncertainty in estimating the mean value.

CI for $E(y)$ vs. PI for y - similarities

- For a given data set, the error in estimating $E(y)$ and $\hat{y}$ grows as $x^\star$ moves away from $\bar{x}$. Thus, the further $x^\star$ is from $\bar{x}$, the wider the confidence and prediction intervals will be.
- If any of the conditions underlying the model are violated, then the confidence intervals and prediction intervals may be invalid as well. This is why it's so important to check the conditions by examining the residuals, etc.

Confidence and prediction intervals for MLR

- In the case of multiple linear regression (regression with many predictors), confidence and prediction intervals for a new prediction works exactly the same way.
- However the formulas are much more complicated since we no longer have just one x, but instead many xs.
- For confidence and prediction intervals for MLR we will focus on the concepts and leave the calculations up to R.

Prediction of evaluation my evaluation score

```
# fit a model
m = lm(score ~ rank + gender + language + cls_perc_eval + cls_students, data = evals)

# create a data frame with the new observation (mine)
prof = data.frame(rank = "teaching", gender = "female",
    language = "english", cls_perc_eval = 90, cls_students = 100)
```

```
# prediction interval
predict(m , prof , interval = "prediction")
```

```
       fit      lwr      upr
 4.337951 3.301877 5.374026
```

Based on this model, we are 95% confident that the predicted evaluation score for this professor is between 3.30 and 5.37.

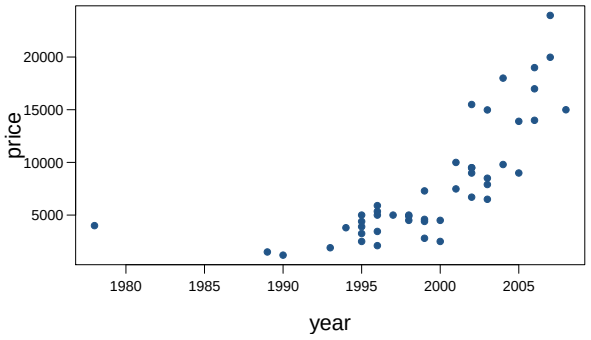
```
# confidence interval
predict(m , prof , interval = "confidence")
```

```
      fit     lwr      upr
 4.337951 4.20273 4.473172
```

Based on this model, we are 95% confident that the predicted evaluation score for a group of professors who share these characteristics is between 4.20 and 4.47.

Truck prices

The scatterplot below shows the relationship between year and price of a random sample of 43 pickup trucks. Describe the relationship between these two variables.

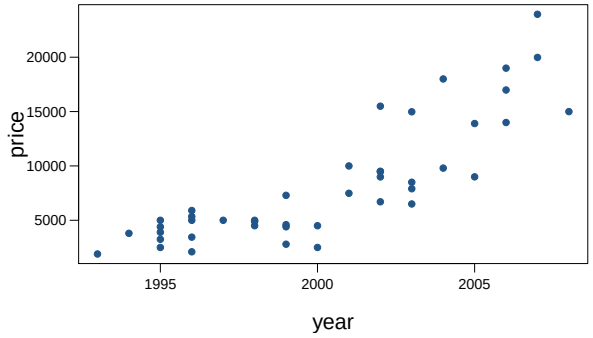


From: *http://faculty.chicagobooth.edu/robert.gramacy/teaching.html*

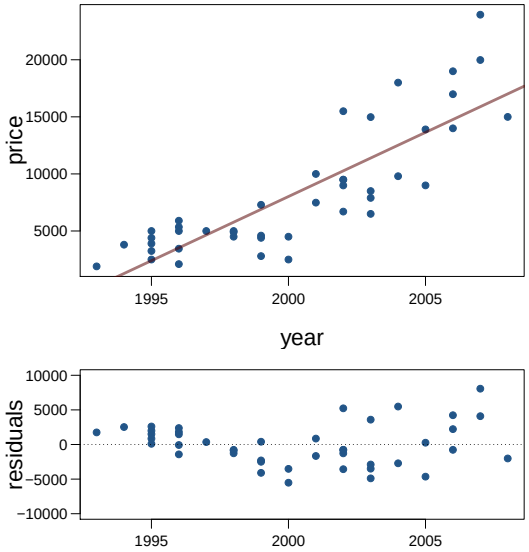
Remove unusual observations

Let's remove trucks older than 20 years, and only focus on trucks made in 1992 or later.

Now what can you say about the relationship?



Truck prices - linear model?

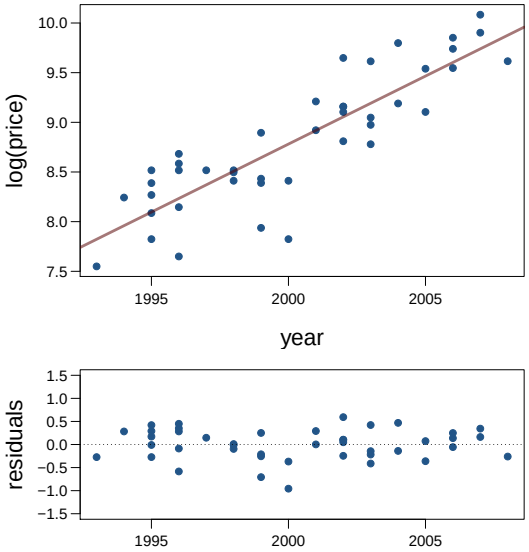


Model:

$$\widehat{price} = b_0 + b_1\ year$$

The linear model doesn't appear to be a good fit since the residuals have non-constant variance.

Truck prices - log transform of the response variable



Model:

$$\widehat{log(price)} = b_0 + b_1\ year$$

We applied a log transformation to the response variable. The relationship now seems linear, and the residuals no longer have non-constant variance.

Interpreting models with log transformation

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-265.07	25.04	-10.59	0.00
pu$year	0.14	0.01	10.94	0.00

Model: $\widehat{log(price)} = -265.07 + 0.14\ year$

- For each additional year the car is newer (for each year decrease in car's age) we would expect the log price of the car to increase on average by 0.14 log dollars.
- which is not very useful...

Working with logs

- Subtraction and logs: $log(a) - log(b) = log(\frac{a}{b})$
- Natural logarithm: $e^{log(x)} = x$
- We can these identities to "undo" the log transformation

Interpreting models with log transformation (cont.)

The slope coefficient for the log transformed model is 0.14, meaning the log price difference between cars that are one year apart is predicted to be 0.14 log dollars.

$$\begin{aligned} \log(\text{price at year } x+1) - \log(\text{price at year } x) &= 0.14 \\ log\left(\frac{\text{price at year } x+1}{\text{price at year } x}\right) &= 0.14 \\ e^{log\left(\frac{\text{price at year } x+1}{\text{price at year } x}\right)} &= e^{0.14} \\ \frac{\text{price at year } x+1}{\text{price at year } x} &= 1.15 \end{aligned}$$

For each additional year the car is newer (for each year decrease in car's age) we would expect the price of the car to increase on average *by a factor of 1.15*.

Recap: dealing with non-constant variance

- Non-constant variance is one of the most common model violations, however it is usually fixable by transforming the response (y) variable
- The most common variance stabilizing transform is the log transformation: $log(y)$, especially useful when the response variable is (extremely) right skewed.
- When using a log transformation on the response variable the interpretation of the slope changes:
  - For each unit increase in x, y is expected on average to decrease/increase by a factor of e^{b_1}.
- Another useful transformation is the square root: $\sqrt{y}$, especially useful when the response variable is counts.
- These transformations may also be useful when the relationship is non-linear, but in those cases a polynomial regression may also be needed.