Final Examination

STA 711: Probability & Measure Theory

Wednesday, 2012 Dec 12, 9:00 am – 12:00n

This is a closed-book examination. You may use a sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing, *please* ask me to clarify it.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets and a pdf/pmf sheet at the end of the test. It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find. Good luck.

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
	/100		/100
Total:	/200		

Print Name: ______________________________

Problem 1. Let $\mathcal{A}$ be a collection of subsets of a nonempty set Ω such that

a. $\Omega \in \mathcal{A}$

b. $A, B \in \mathcal{A} \Rightarrow A \backslash B = A \cap B^c \in \mathcal{A}$.

a) (8) Prove that $\mathcal{A}$ is a field.

b) (12) Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{B} = \{B \subset \Omega : \#(B) \text{ is even}\}$, the sets with 0, 2, or 4 elements. Show that $\mathcal{B}$ is a λ-system. Is it a field?
◯ Yes ◯ No Why?

Problem 2: For $0 < p < 1$ let $\{X_i : i \in \mathbb{N}\} \overset{\text{iid}}{\sim} \mathsf{Ge}(p)$ be iid with the geometric probability distribution with probability mass function (pmf)

$$\mathsf{P}[X_i = k] = pq^k, \quad k \in \mathbb{N}_0 \equiv \{0, 1, 2, \cdots\}, \quad q \equiv (1 - p).$$

a) (8) Find[1] the pmf for $Y_n \equiv \max_{1 \le i \le n} X_i$:

b) (8) Find[2] the pmf for $Z_n \equiv \min_{1 \le i \le n} X_i$:

c) (4) Find the chf (Characteristic Function) for $S_n \equiv \sum_{1 \le i \le n} X_i$:

[1] Suggestion: Find the CDFs for X_i and then for Y_n first.
[2] What's the probability that Z_n is *greater* than z?

Problem 3: The random variables $\{X_i\}$ are all independent and all satisfy $\mathsf{E}[X_i{}^4] \le 1.0$, but they may have different distributions. Let $S_n \equiv \sum_{i=1}^n X_i$ be their partial sum.

a) (8) Does it follow without any further assumptions that S_n/n converges almost surely? ◯ Yes ◯ No Give a proof or counter-example.

b) (8) If in addition we know $X_n \to 0$ in probability, for which (if any) $0 < p < \infty$ does it follow that $X_n \to 0$ in L_p? Why?

c) (4) Give the best bound you can: (+1xc for showing it's best possible)

$\mathsf{P}[X_1 \ge 2] \le$ ____________________

Problem 4: Let $\Omega = \mathbb{R}_+ = [0, \infty)$ be the positive half-line, with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure P given by $\mathsf{P}(d\omega) = e^{-\omega}\, d\omega$ or, equivalently,

$$\mathsf{P}\big[(a, b]\big] = e^{-a} - e^{-b} \qquad 0 \le a \le b < \infty.$$

For each integer $n \in \mathbb{N} = \{1, 2, \cdots\}$ define a random variable on $(\Omega, \mathcal{F})$ by

$$X_n(\omega) := \begin{cases} 0 & \text{if } \omega < n \\ 1 & \text{if } \omega \ge n \end{cases}$$

a) (4) Find the mean $m_n = \mathsf{E}[X_n]$ for each $n \in \mathbb{N}$ and the covariance $\Sigma_{mn} = \mathsf{E}[(X_m - m_m)(X_n - m_n)]$ for each $m \le n \in \mathbb{N}$:

$m_n =$ $\qquad\qquad\qquad\qquad$ $\Sigma_{mn} =$

b) (4) Give the probability distribution *measure* $\mu_n(\cdot)$ of X_n for each n:

Problem 4 (cont'd): As before, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $\mathsf{P}(d\omega) = e^{-\omega}\, d\omega$, and $X_n(\omega) := \mathbf{1}_{[n,\infty)}(\omega)$ for $n \in \mathbb{N}$ (see footnote[3])

c) (4) For each fixed $n \in \mathbb{N}$ give the σ-algebra $\sigma(X_n)$ explicitly:

$$\sigma(X_n) = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

d) (4) Does the σ-algebra $\mathcal{G} = \sigma(X_1, X_2, ...)$ generated by all the X_n's contain *all* the Borel sets in $\mathbb{R}_+$? ◯ Yes ◯ No If so, say why; if not, find a Borel set $B \in \mathcal{F}$ that is *not* in $\mathcal{G}$.

e) (4) Are X_1 and X_2 independent? ◯ Yes ◯ No Justify your answer.

[3]Recall that the *indicator* random variable $\mathbf{1}_A(\omega)$ is one if $\omega \in A$, otherwise zero.

Problem 5: As in Problem 4, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $\mathsf{P}(d\omega) = e^{-\omega}\, d\omega$, and $X_n(\omega) := \mathbf{1}_{[n,\infty)}(\omega)$ for $n \in \mathbb{N}$.

a) (4) Prove that the partial sums $S_n := \sum_{j=1}^n X_j$ converge almost surely as $n \to \infty$ to some limiting random variable $S \equiv \sum_{j=1}^\infty X_j$.

b) (4) Do the partial sums $S_n := X_1 + \cdots + X_n$ converge to S in L_1 as $n \to \infty$? ◯ Yes ◯ No Justify your answer.

c) (4) Give the name[4] and the mean of the probability distribution of the limit $S = \sum_{j=1}^\infty X_j$.

d) (8) Set $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$, the σ-algebra generated by the first n of the X_k's. Find the indicated conditional expectations:

$\mathsf{E}[X_4 \mid \mathcal{F}_2] =$ $\qquad$ $\mathsf{E}[S \mid \mathcal{F}_2] =$

[4] Remember, there's a list of distributions with names and means at the back of this exam. Exactly what must ω be to make $S = 0$? $S = 2$? $S = k$?

Problem 6: Be specific for each of the following, leaving no parameters unspecified, but no need to prove convergence. For each part you may specify either the distributions themselves $\mu_n(dx)$, $\mu(dx)$ or random variables X_n, X with those distributions.

a) (8) Give an example of a sequence of discrete distributions that converge in distribution to an absolutely-continuous distribution.

b) (8) Give an example of a sequence of absolutely-continuous distributions that converge in distribution to a discrete distribution.

c) (4) Give an example of a distribution supported on only rational values (so $\mu_n(B) = 1$ for some closed set $B \subset \mathbb{Q}$) that converges to one supported on only irrational values (so $\mu(B) = 1$ for some closed $B \subset \mathbb{Q}^c$).

Problem 7: Let $\{X_j\}_{1\le j\le 3}$ be independent random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ representing the outcomes on three independent fair 6-sided dice.

a) (6) How many points must Ω have, at minimum? Why?

b) (6) Is is possible to find iid X_1, X_2, X_3 each uniform on $\{1,2,3,4,5,6\}$ on the space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0,1]$ and $\mathsf{P} = d\omega$ Lebesgue measure on the Borel sets $\mathcal{F} = \mathcal{B}$? ◯ Yes ◯ No If so, give a possible version of $X_1 : \Omega \to \mathbb{R}$ (+1xc for all three, X_1, X_2, X_3); if not, why?

c) (8) Let $Y = X_1 + X_2$ and $Z = X_2 + X_3$. Find[5]: $\mathsf{E}[Y \mid Z] =$

[5]Suggestion: First find $\mathsf{E}[X_1 \mid X_2, X_3]$ and $\mathsf{E}[X_2 \mid Z \equiv X_2 + X_3]$.

Problem 8: Let $X_j \overset{\text{iid}}{\sim} \mathsf{Po}(1)$ be independent random variables, all with the unit-mean Poisson distribution.

a) (8) Find the characteristic function $\phi(\omega) = \mathsf{E}\left[e^{i\omega X_j}\right]$ of X_j and the log chf $\psi(\omega) \equiv \log \phi(\omega)$.

b) (6) For numbers $a > 0$, find the log characteristic function $\psi_1(\omega)$ of $(X_j - 1)/a$.

c) (6) Let $S_n = X_1 + \cdots + X_n$ be the partial sum. Find a sequence $a_n > 0$ such that the log characteristic function $\psi_n(\omega)$ of $(S_n - n)/a_n$ converges to $-\omega^2/2$ for every ω, and explain what this says about the limiting probability distribution of S_n (*i.e.*, about the $\mathsf{Po}(n)$ distribution for large n).[6]

[6] Recall the Taylor series $e^x = 1 + x + x^2/2 + o(x^2) \approx 1 + x + x^2/2$ near $x \approx 0$.

Problem 9: The random variables X and Y have a distribution generated by the following mechanism. A fair coin is tossed; if it falls Heads, then $X = Y = 0$; if it falls Tails, then X and Y are drawn independently from the standard normal $\mathsf{No}(0,1)$ distribution with CDF $\Phi(z) \equiv \int_{-\infty}^{z} e^{-t^2/2} dt/\sqrt{2\pi}$.

a) (4) Are X and Y independent? ◯ Yes ◯ No Why?

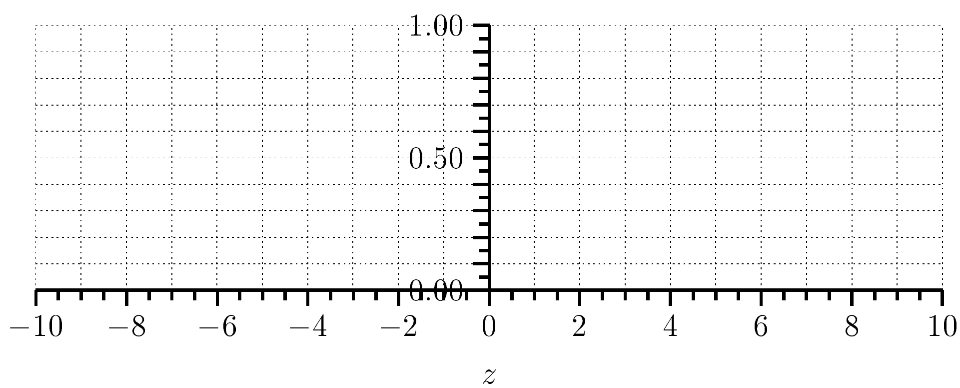
b) (4) Set $Z \equiv 3X + 4Y$. If the coin falls Tails (in which case $X, Y \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$), find the conditional CDF for Z (you may use $\Phi(\cdot)$ in your expression):

$\mathsf{P}[Z \le z \mid \text{Tails}] =$

c) (6) Now find the unconditional CDF for $Z = 3X + 4Y$:

$$\mathsf{P}[Z \le z] = \begin{cases} & \\ & \\ & \end{cases}$$

and sketch a **very very rough** plot of it:



Problem 9 (cont'd):

d) (6) Let $\mathcal{G}$ be the σ-algebra generated by Z. Find the conditional expectation of X, given $\mathcal{G} = \sigma(Z)$:

$\mathsf{E}[X \mid \mathcal{G}] =$ ______________

Problem 10: Let $\{X_n > 0\}$ and $X > 0$ be positive random variables with $X_n \to X$ *a.s.* Choose True or False below; no need to explain (unless you can't resist). Each is 3pt, except g) (just 2pt, and difficult to get wrong).

a) T F $\log(X_n) \to \log(X)$ in probability.

b) T F $X_n \to X$ in L_2 if each $\mathsf{E}\,[|X_n|^3] \le \pi$.

c) T F $\log(X_n) \to \log(X)$ in L_1 if each $\mathsf{E}[|X_n|^3] \le \pi$.

d) T F $\limsup_{n\to\infty} \mathsf{E}[\log(1+X_n)] \ge \mathsf{E}[\log(1+X)]$.

e) T F $X \in L_2$ if, for some $t > 0$, $\mathsf{E}[\exp(t \cdot X)] < \infty$.

f) T F $X \in L_2$ if, for some $t < 0$, $\mathsf{E}[\exp(t \cdot X)] < \infty$.

g) T F The answer you chose for this question is T.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	