Final Examination

STA 711: Probability & Measure Theory

Tuesday, 2013 Dec 10, 9:00 am – 12:00n

This is a closed-book examination. You may use a sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing, *please* ask me to clarify it.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets and a pdf/pmf sheet at the end of the test. It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find. Good luck.

Print Name: ______________________________

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
	/80		/80
Total:	/160		

Problem 1: Let $\{\mathcal{L}_n\}$ and $\{\mathcal{P}_n\}$ be nested sequences of λ-systems and π-systems[1], respectively, on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, *i.e.*, that satisfy $\mathcal{L}_n \subset \mathcal{L}_{n+1} \subset \mathcal{F}$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \mathcal{F}$ for each $n \in \mathbb{N}$. For each of parts a)–d), give a proof or a counter-example.

a) (5) Is $\cap\mathcal{L}_n$ a λ-system? ◯ Yes ◯ No Why?

b) (5) Is $\cup\mathcal{L}_n$ a λ-system? ◯ Yes ◯ No Why?

[1] Recall: $\mathcal{L}$ is a λ-system if (a) $\Omega \in \mathcal{L}$, (b) $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$, and (c) $\{A_n\} \subset \mathcal{L}$ disjoint $\Rightarrow \cup A_n \in \mathcal{L}$. $\mathcal{P}$ is a π-system if $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$.

Problem 1 (cont'd): Still $\mathcal{L}_n \subset \mathcal{L}_{n+1} \subset \mathcal{F}$ are λ-systems and $\mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \mathcal{F}$ are π-systems on $(\Omega, \mathcal{F}, \mathsf{P})$.

c) (5) Is $\cap \mathcal{P}_n$ a π-system? ◯ Yes ◯ No Why?

d) (5) Is $\cup \mathcal{P}_n$ a π-system? ◯ Yes ◯ No Why?

Problem 2: Let $\Omega = (0,1]^2$ be the unit square, $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel sets, and P Lebesgue measure. Consider several families of events:[2]

$$\mathcal{A} := \{(0,a] \times (0,a] : \ 0 < a \le 1\} \qquad \mathcal{B} := \{(0,c] \times (c,1] : \ 0 < c \le 1\}$$
$$\mathcal{C} := \{(0,a] \times (0,b] : \ 0 < a \le b \le 1\} \quad \mathcal{D} := \{(0,a] \times (0,b] : \ 0 < a, b \le 1\}$$

a) (4) Which (if any) of these is a π-system? Circle those that are:

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

b) (4) Which (if any) of these generates $\mathcal{F}$, *i.e.*, has $\sigma(\cdots) = \mathcal{F}$?

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

c) (6) Prove both your assertions about $\mathcal{A}$.

d) (6) Prove both your assertions about $\mathcal{C}$.[3]

[2] Drawing pictures helps.

[3] I wouldn't suggest spending lots of time on the "b)" section of this part until the rest of the exam was mostly done.

Problem 3: Let $\Omega = \mathbb{R}_+ = [0, \infty)$ be the positive half-line, with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure P given by $\mathsf{P}(d\omega) = 2e^{-2\omega}\,d\omega$ or, equivalently,

$$\mathsf{P}\big[(a, b]\big] = e^{-2a} - e^{-2b} \qquad 0 \le a \le b < \infty.$$

For each integer $n \in \mathbb{N} = \{1, 2, \cdots\}$ define a random variable on $(\Omega, \mathcal{F})$ by

$$X_n(\omega) := \omega^n.$$

a) (4) Find the mean $m_n = \mathsf{E}[X_n]$ for each $n \in \mathbb{N}$ and the covariance $\Sigma_{mn} = \mathsf{E}[(X_m - m_m)(X_n - m_n)]$ for each $m, n \in \mathbb{N}$ (see footnote[4]):

$m_n =$ $\qquad\qquad\qquad\qquad$ $\Sigma_{mn} =$

b) (4) Give the distribution $\mu_n(\cdot)$ of X_n, a probability measure on $(\mathbb{R}, \mathcal{B})$:

$\mu_n(B) =$

[4] Recall $\Gamma(\alpha) := \int_0^\infty z^{\alpha-1}e^{-z}\,dz$ for $\alpha > 0$, or $(\alpha - 1)!$ for $\alpha \in \mathbb{N}$

Problem 3 (cont'd): As before, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $\mathsf{P}(d\omega) = 2e^{-2\omega}\, d\omega$, and $X_n(\omega) := \omega^n$ for $n \in \mathbb{N}$

c) (4) Find the indicated conditional expectation. Explain your answer.

$\mathsf{E}[X_2 \mid X_4](\omega) =$ ____________________

d) (4) Does X_n converge to a limit $X \in L_2$? If so, find X; if not, why?
◯ Yes ◯ No

e) (4) For which (if any) $p > 0$ does $Y_n = \sum_{j=0}^{n} X_j/j!$ converge in L_p as $n \to \infty$ to a limit $Y \in L_p$? Why?

Problem 4: Let $\{A_n\} \subset \mathcal{F}$ and $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ with $\|X_n\|_1 \le 1$ and $P(A_n) \to 0$.

a) (8) Does it follow that $\mathsf{E}X_n\mathbf{1}_{A_n} \to 0$? ◯ Yes ◯ No Prove it, or find a counter-example:

b) (8) Does it follow that $\mathsf{E}X_1\mathbf{1}_{A_n} \to 0$? ◯ Yes ◯ No Prove it, or find a counter-example:

c) (4) Would either of your answers to a) or b) change if we have $\{X_n\} \subset L_2(\Omega, \mathcal{F}, \mathsf{P})$ with $\|X_n\|_2 \le 1$? Explain.

Problem 5: Let $\{A_n\}$ be events on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{P}(A_n) = 2^{-n}$ and for $n \geq 0$ set

$$X_n := 2^n \mathbf{1}_{A_n}$$

a) (4) Show that $Y := \sum_{n=0}^{\infty} X_n$ is finite almost-surely:

b) (4) For which $0 < p \leq \infty$ is $X_n \in L_p$? Why?

c) (6) For which $0 < p \leq \infty$ is $Y \in L_p$? Why?

d) (6) Is the collection $\{X_n\}$ Uniformly Integrable? ◯ Yes ◯ No
Why?

Problem 6: Let $X \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ have mean $\mathsf{E}X = 0$ and variance $\sigma^2 = \mathsf{E}X^2$.

a) (8) For all $a > 0$ and $t \in \mathbb{R}$ prove the one-sided bound

$$\mathsf{P}[X > a] \le \frac{\sigma^2 + t^2}{(a+t)^2}$$

b) (8) Find the value of $t > 0$ that minimizes this bound (Hint: logarithms make this easier). Simplify!

c) (4) Find the resulting bound on tail probabilities (Simplify!):
$\mathsf{P}[X > a] \le$

Problem 7: The random variables X and Z are independent, with distributions

$$X \sim \mathsf{No}(0,1) \qquad \mathsf{P}[Z=+1] = 1/2 = \mathsf{P}[Z=-1]$$

while $Y := XZ$ is their product. Simplify all answers.

a) (6) What is the probability distribution of Y?

b) (4) What is the covariance of X and Y?

c) (5) Are X and Y independent? ◯ Yes ◯ No Why?

d) (5) Are Y and Z independent? ◯ Yes ◯ No Why?

Problem 8: Let $\{X_n\} \subset L_2(\Omega, \mathcal{F}, \mathsf{P})$ be independent and identically distributed with mean $\mu = \mathsf{E}X_1$ and variance $\sigma^2 = \mathsf{E}(X_1 - \mu)^2$, and let $\mathcal{F}_n = \sigma\{X_j : \ 1 \le j \le n\}$. Fix any $\epsilon > 0$. Choose True or False below; no need to explain (unless you can't resist). Each is 2pt.

a) T F For any $Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, $M_n := \mathsf{E}[Y \mid \mathcal{F}_n]$ is a martingale.

b) T F For Y and $M_n := \mathsf{E}[Y \mid \mathcal{F}_n]$ as above, $|M_n| \le |Y|$ a.s.

c) T F If Z is measurable over $\cap_{n \ge m} \mathcal{F}_n$ for each $m \in \mathbb{N}$, then Z is constant a.s.

d) T F Almost surely, $\{n : \ X_n > n\epsilon\}$ is a finite set.

e) T F $\int_{-\epsilon}^{\epsilon} |x|^{-1/2}\, dx$ is well-defined and finite.

f) T F If $\mathsf{E}[\exp(Z)] < \infty$ and $\mathsf{E}[\exp(-Z)] < \infty$, then $Z \in L_2$.

g) T F Any function of $\{X_{2n} : n \in \mathbb{N}\}$ is independent of any function of $\{X_{2n+1} : n \in \mathbb{N}\}$.

h) T F $\mathsf{E}[X_1 \mid \ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6) = 42] = 7$

i) T F If $Y_n := X_n^2$ then $\{Y_n\}$ are UI

j) T F Necessarily $\mathsf{P}\big[\limsup |X_n| = \infty\big] = 1 = \mathsf{P}\big[\liminf |X_n| = 0\big]$

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	