

Sta 711: Homework 5

Convergence

1. Let X be a nonnegative random variable. Show that:

- (a) $\lim_{n \rightarrow \infty} n \mathbf{E}(\frac{1}{X} \mathbf{1}_{[X > n]}) = 0$.
- (b) $\lim_{n \rightarrow \infty} n^{-1} \mathbf{E}(\frac{1}{X} \mathbf{1}_{[X > n^{-1}]}) = 0$.

2. Let $X \sim \text{Un}(0, 4]$ be uniformly distributed on the interval $(0, 4]$, and set $Y := 1/X$ and $Z := \log(4Y)$. Suggestion: First find out what is the distribution of Z , by computing $\mathbf{P}[Z > z]$ for $z \in \mathbb{R}$.

- (a) What bound does Markov's inequality give for $\mathbf{P}[X > 3]$?
- (b) What bound does Chebychev's inequality give for $\mathbf{P}[|X - 2| > 1]$?
- (c) What bound does Markov's inequality give for $\mathbf{P}[Y > 1]$?
- (d) What bound does Markov's inequality give for $\mathbf{P}[Z > 2]$?
- (e) What are the exact values of $\mathbf{P}[X > 3]$, $\mathbf{P}[|X - 2| > 1]$, $\mathbf{P}[Y > 1]$, and $\mathbf{P}[Z > 2]$?

3. Let A and B be events in $(\Omega, \mathcal{F}, \mathbf{P})$ with probabilities $a = \mathbf{P}(A)$ and $b = \mathbf{P}(B)$ respectively. Show that $\mathbf{P}(A \cap B) \leq \sqrt{ab}$.

4. Suppose $\{X_n\}, X$ are real valued RVs defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that $X_n(\omega) \rightarrow X(\omega)$ for all $\omega \in \Omega$. Show that for every $\epsilon > 0$, there is an event Λ_ϵ with $\mathbf{P}(\Lambda_\epsilon) < \epsilon$ and

$$\sup_{\omega \in \Lambda_\epsilon^c} |X(\omega) - X_n(\omega)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus the convergence is uniform except on an arbitrarily small set. (For more on this result, called Egorov's Theorem, see page 89 of the text.)

5. For a random variable X , $1 < p < q < \infty$, show¹ that

$$0 \leq \|X\|_1 \leq \|X\|_p \leq \|X\|_q \leq \|X\|_\infty$$

6. For $1 < p < q < \infty$, show that

$$L_\infty \subset L_q \subset L_p \subset L_1$$

where $L_p = \{X : \|X\|_p < \infty\}$.

¹Hint: Jensen's inequality may help for some parts

7. The “moment generating function” of a real-valued random variable X (or of its distribution) is the extended real-valued function $M_X(t) := \mathbf{E} \exp(tX)$ of $t \in \mathbb{R}$. Show that a nonnegative random variable $X \geq 0$ is in L_1 if $M_X(t) < \infty$ for any $t > 0$.
8. Show that Minkowski’s Inequality fails for $0 < p < 1$ — *i.e.*, find $(\Omega, \mathcal{F}, \mathbf{P})$ and $X, Y \in L_p(\Omega, \mathcal{F}, \mathbf{P})$ for which $\|X + Y\|_p > \|X\|_p + \|Y\|_p$ for some $0 < p < 1$.