Sta 711: Homework 8

Convergence Of Series, Strong Law

1. For $n \in \mathbb{N}_0$ let $j = \lfloor \log_2 n \rfloor$ and $i = n - 2^j$, so $n = i + 2^j$ with $j \in \mathbb{N}_0$ and $0 \le i < 2^j$. For $\omega \in \Omega = (0, 1]$ set
$$X_n(\omega) := n \mathbf{1}_{\{i/2^j < \omega \le (i+1)/2^j\}}$$
Verify that X_n converges *pr.* but not *a.s* (for Lebesgue measure P on the Borel sets $\mathcal{F}$), and find an explicit subsequence that converges almost-surely. What is the limit? Does this subsequence converge in L_1?

2. One version of the SLLN states that if $\{X_n, n \ge 1\}$ are iid with $\mathsf{E}|X_1| < \infty$, then $S_n/n \to \mathsf{E}(X_1)$ a.s. Show that also
$$S_n/n \to \mathsf{E}(X_1) \qquad \text{in } L_1.$$

3. Define a sequence $\{X_n\}$ of random variables iteratively as follows. Let $X_0 \equiv c > 0$ be any positive constant and, for $n \in \mathbb{N}$, let X_n have a uniform distribution on $(0, X_{n-1}]$ (independent of $\{X_j : \ j < n - 1\}$). Show that
$$\frac{1}{n} \log X_n$$
converges a.s. and find the almost sure limit.

Two Fun Concepts

4. Let f_0 and f_1 be probability mass functions (pmfs) on the set $\mathcal{S} := \{1, 2, ..., 100\}$, i.e., nonnegative functions satisfying $\sum_{y \in \mathcal{S}} f_\theta(y) = 1$ for $\theta = 0, 1$. Let $\{X_n\}$ be iid random variables with pmf f_0, so $\mathsf{P}[X_n = y] = f_0(y)$ for $y \in \mathcal{S}$. Set
$$\Lambda_n := \prod_{i=1}^{n} \frac{f_1(X_i)}{f_0(X_i)}$$
Prove that $\Lambda_n \to 0$ almost surely if $f_0(y) \ne f_1(y)$ for at least one $y \in \mathcal{S}$. Be careful about any points where $f_0(y) = 0$ or $f_1(y) = 0$. The quantity Λ_n is called the *Bayes factor* or *likelihood ratio* against the "null hypothesis" f_0. This shows that the Likelihood Ratio Test always succeeds (eventually!).[1]

5. Suppose $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is measurable and Lebesgue integrable. Let $\{X_n, n \ge 1\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be standard exponential random variables with pdf $f(x) = e^{-x} \mathbf{1}_{\{x>0\}}$ and define $Y_n := g(X_n) \exp(X_n)$. What is the limit as $n \to \infty$ of $\overline{Y}_n = \sum_{i=1}^{n} Y_i / n$? In what sense, and why? If $g \in L_2(\mathbb{R}_+, e^x \, dx)$, find the variance of $\overline{Y}_n$ and show that it converges to zero as $n \to \infty$. Show how this lets us estimate $\int_{\mathbb{R}_+} g(x) \, dx$.

[1] Hint: $(\forall x \in \mathbb{R})$ $e^x \ge 1 + x$ (with equality only at $x = 0$), and so $(\forall y > 0)$ $\log y \le y - 1$. Or, for another approach, apply Jensen's inequality to the convex function $-\log y$ on $\mathbb{R}_+$.