Sta 711: Homework 10

Conditional Expectation

1. Let $\{N_t\}_{t\geq 0}$ be a homogeneous Poisson process with rate λ, so $N_0 = 0$ and for every $n \in \mathbb{N}$ and $0 = t_0 < t_1 < ... < t_n < \infty$ the random variables $X_i := [N_{t_i} - N_{t_{i-1}}]$ for $1 \leq i \leq n$ are independent with marginal distributions $X_i \sim \mathsf{Po}\big(\lambda(t_i - t_{i-1})\big)$. For $0 < s < t < \infty$ find the conditional expectations:

$$\mathsf{E}[N_s \mid N_t] = \qquad\qquad \mathsf{E}[N_t \mid N_s] =$$

2. Let $\{X_1, X_2\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be iid unit-rate exponential random variables, $t > 0$. Find:

   (a) $\mathsf{E}[X_1 \mid X_1 + X_2] =$

   (b) $\mathsf{P}[X_1 < 3 \mid X_1 + X_2] =$

   (c) $\mathsf{E}[X_1 \mid X_1 \wedge t] =$

   (d) $\mathsf{E}[X_1 \mid X_1 \vee t] =$

3. Let $X, Y \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ and suppose $\mathsf{E}[X \mid Y] = \phi(Y)$ for a monotonically decreasing Borel function $\phi : \mathbb{R} \to \mathbb{R}$. Prove that $\mathsf{Cov}(X, Y) \leq 0$.

4. If $X \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, show

$$\mathsf{E}\Big[\big(X - \mathsf{E}[X \mid \mathcal{H}]\big)^2\Big] \geq \mathsf{E}\Big[\big(X - \mathsf{E}[X \mid \mathcal{G}]\big)^2\Big]$$

   What does this imply for the trivial σ-algebra $\mathcal{H} = \{\emptyset, \Omega\}$?

Martingales

A sequence $\{(X_n, \mathcal{F}_n),\ n \geq 0\}$ of random variables $X_n \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ and a nested sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$ is a *martingale* if for each $0 \leq n \leq m < \infty$

$$X_n = \mathsf{E}[X_m \mid \mathcal{F}_n],$$

so on average the sequence neither increases nor decreases. Note this implies that X_n is $\mathcal{F}_n$-measurable. By the tower property of conditional expectations, it is enough to check this condition for $m = n + 1$.

5. A sequence $\{(X_n, \mathcal{F}_n),\ n \geq 0\}$ is *predictable* if X_{n+1} is $\mathcal{F}_n$-measurable for each n. Show every predictable martingale is constant (*i.e.*, $X_n = X_0$ a.s.).

6. A sequence $\{(X_n, \mathcal{F}_n),\ n \ge 0\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ is a *submartingale* if $\mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F}$ and $X_n \le \mathsf{E}[X_m \mid \mathcal{F}_n]$ for each $0 \le n \le m < \infty$.

   Let $\{(X_n, \mathcal{F}_n),\ n \ge 0\}$ and $\{(Y_n, \mathcal{F}_n),\ n \ge 0\}$ be submartingales on $(\Omega, \mathcal{F}, \mathsf{P})$. Show that their max $(X_n \vee Y_n)$ and sum $(X_n + Y_n)$ are submartingales too.

7. Fix $0 < p < 1$, set $q := 1 - p$, and let $\{\xi_j\}$ be iid random variables with $\mathsf{P}[\xi_j = 1] = p$ and $\mathsf{P}[\xi_j = -1] = q$. Set

$$S_n := \sum_{j \le n} \xi_j,$$

   a random walk (possibly an asymmetric one) on the integers starting at $S_0 = 0$.

   (a) For which $\alpha \in \mathbb{R}$ is $X_n := [S_n - \alpha n]$ a martingale?

   (b) For which $\alpha, \beta \in \mathbb{R}$ is $Y_n := [(S_n - \alpha n)^2 - \beta n]$ a martingale?

   (c) For which $r > 0$ is $Z_n := [r^{S_n}]$ a martingale?

   Of course the answers will depend on p.