# Midterm Examination II

STA 711: Probability & Measure Theory

Thursday, 2013 Nov 14, 11:45 am – 1:00pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing, *please* ask me to clarify it.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find. For full credit, give answers in **closed form** (no unevaluated sums, integrals, maxima, *etc.*) where possible and **simplify**.
Good luck!

| | |
|---|---|
| 1. | /20 |
| 2. | /20 |
| 3. | /20 |
| 4. | /20 |
| 5. | /20 |
| Total: | /100 |

Print Name: ______________________________

**Problem 1**: Let $\{X_i\} \subset L_2(\Omega, \mathcal{F}, \mathsf{P})$ be an uncorrelated sequence of mean zero random variables that is uniformly bounded in $L_2$, *i.e.*,

$$(\forall i \neq j) \quad \mathsf{E}[X_i \, X_j] = 0 \qquad\qquad (\exists c < \infty)(\forall i) \quad \mathsf{E}|X_i|^2 \leq c$$

Show that for any $\alpha > \frac{1}{2}$

$$n^{-\alpha} \sum_{j=1}^{n} X_j \to 0$$

in $L_2$ as $n \to \infty$ and, in particular, $\bar{X}_n \to 0$ in $L_2$.

**Problem 2**: Let $\Omega = \mathbb{R}_+$ with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure

$$\mathsf{P}[A] := \int_A e^{-\omega}\, d\omega$$

for $A \in \mathcal{F}$. For $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ set $X_n(\omega) := 2^n\, \mathbf{1}_{[n,\infty)}(\omega)$.

a) (6) Find the $L_p$ norm of $X_n$ for each $1 \le p \le \infty$:
$\|X_n\|_p =$

b) (6) In which sense(s) does $X_n \to 0$ as $n \to \infty$? Justify each answer.
◯ $a.s.$ ◯ $pr.$ ◯ $L_1$ ◯ $L_2$ ◯ $L_\infty$ ◯ in dist.

c) (8) Set $Z := \sum_{0 \le n < \infty} X_n$. Evaluate $Z(\omega)$ explicitly and find its mean. For full credit, answer must be closed-form and you must **simplify**.

$Z(\omega) =$ ____________ $\mathsf{E}[Z] =$ ____________

**Problem 3**: Let $\Omega := \{1,2,3,4,5\}$, $\mathcal{F} := 2^\Omega$, and $\mathsf{P}(A) := \sum_{\omega\in A} \frac{\omega}{15}$ for $A \in \mathcal{F}$. Set $X(\omega) := \omega$ for $\omega \in \Omega$ and $B := \{1,2\} \in \mathcal{F}$.

a) (2) Find: $\mathsf{E}[X] =$ $\mathsf{P}[B] =$

b) (4) Find $Y_1 := \mathsf{E}[X \mid \mathcal{G}_1]$ for $\mathcal{G}_1 := \big\{\emptyset, B, B^c, \Omega\big\}$.
$Y_1(\omega) =$

c) (4) Find $Y_2 := \mathsf{E}[X \mid \mathcal{G}_2]$ for $\mathcal{G}_2 := \big\{\emptyset, \Omega\big\}$.
$Y_2(\omega) =$

**Problem 3 (cont'd)**: Now let $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P}(A) := \int_A 2\omega \, d\omega$ for $A \in \mathcal{F}$. Let $X(\omega) := \omega$ and $Y(\omega) := \mathbf{1}_{\{\omega > 1/2\}}$.

d) (2) Find: $\mathsf{E}[X] =$ $\qquad\qquad$ $\mathsf{E}[Y] =$

e) (4) Find $Y_3 := \mathsf{E}[X \mid Y]$:
$Y_3(\omega) =$

f) (4) Find $Y_4 := \mathsf{E}[Y \mid X]$:
$Y_4(\omega) =$

**Problem 4**: Let $\{X_i : i \in \mathbb{N}\}$ be iid with $\mathsf{P}[X_i = 1] = \mathsf{P}[X_i = -1] = 1/2$, and set $S_n := \sum_{1 \le i \le n} X_i$, the simple random walk on $\mathbb{Z}$.

a) (6) Find the characteristic function for $X_1$ (Simplify![1]):
$\phi_1(\omega) = \mathsf{E}e^{i\omega X_1} =$

b) (6) Find the characteristic function for $S_n$ (Simplify!):
$\phi_n(\omega) = \mathsf{E}e^{i\omega S_n} =$

[1]Recall $e^{x+iy} = e^x[\cos y + i \sin y]$

**Problem 4 (cont'd)**: Recall that $\{X_i\}$ are iid with $\mathsf{P}[X_i = \pm 1] = 1/2$ and $S_n := \sum_{1 \le i \le n} X_i$.

c) (4) Use derivatives of the characteristic function for $S_n$ to help you find the first two moments of $S_n$:

$\mathsf{E}[S_n] =$ __________________ $\mathsf{E}[S_n^2] =$ __________________

d) (4) Use a second-order Taylor series for $\psi_n(\omega) := \log \phi_n(\omega)$ to verify that $\phi_n(\omega/\sqrt{n}) \to \exp(-\omega^2/2)$ as $n \to \infty$. What does that say about the distribution of $S_n/\sqrt{n}$?

**Problem 5**: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real.

a) T F Hölder's inequality says $\|X+Y\|_1 \le \|X\|_p + \|Y\|_q$ if $\frac{1}{p}+\frac{1}{q}=1$.

b) T F For any RV $Y$ and any $a>0$, $t>0$, $\mathsf{P}[Y \ge a] \le \mathsf{E}[e^{t(Y-a)}]$.

c) T F For any positive RVs $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$,

$$\mathsf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \left[\mathsf{E} X_n\right].$$

d) T F If $X_n \to X$ *pr.* then $\cos(tX_n) \to \cos(tX)$ in $L_1$ for each $t \in \mathbb{R}$.

e) T F For any UI random variables $\{X_\alpha\}$, if each $X_\alpha \in L_2$ then $\{|X_\alpha|^2\}$ is also UI.

f) T F Let $X \sim \mathsf{Po}(1)$ have the Poisson distribution with $\mathsf{P}[X=k] = e^{-1}/k!$ for $k \in \mathbb{Z}_+ = \{0,1,2,...\}$. Then $\mathsf{P}[X \text{ is even}] > 1/2$.

g) T F Two $\sigma$-fields $\mathcal{F}_1$, $\mathcal{F}_2$ are independent if and only if $\mathsf{P}[F_1 \cap F_2] = \mathsf{P}[F_1]\ \mathsf{P}[F_2]$ for all events $F_i \in \mathcal{F}_i$.

h) T F Random variables $X$ and $Y$ are independent if and only if $\mathsf{E}[\mathbf{1}_A(X) \cdot \mathbf{1}_B(Y)] = \mathsf{E}[\mathbf{1}_A(X)] \cdot \mathsf{E}[\mathbf{1}_B(Y)]$ for all Borel sets $A, B \in \mathcal{B}(\mathbb{R})$.

i) T F If $\{X_n\}$ is UI then for some $p>1$ and $B<\infty$ each $\|X_n\|_p \le B$.

**Blank Worksheet**

**Another Blank Worksheet**

| **Name** | **Notation** | **pdf/pmf** | **Range** | **Mean** $\mu$ | **Variance** $\sigma^2$ | |
|---|---|---|---|---|---|---|
| **Beta** | Be$(\alpha, \beta)$ | $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ | $x \in (0,1)$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |
| **Binomial** | Bi$(n, p)$ | $f(x) = \binom{n}{x} p^x q^{(n-x)}$ | $x \in 0, \cdots, n$ | $n\,p$ | $n\,p\,q$ | $(q = 1-p)$ |
| **Exponential** | Ex$(\lambda)$ | $f(x) = \lambda\, e^{-\lambda x}$ | $x \in \mathbb{R}_+$ | $1/\lambda$ | $1/\lambda^2$ | |
| **Gamma** | Ga$(\alpha, \lambda)$ | $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$ | $x \in \mathbb{R}_+$ | $\alpha/\lambda$ | $\alpha/\lambda^2$ | |
| **Geometric** | Ge$(p)$ | $f(x) = p\, q^x$ | $x \in \mathbb{Z}_+$ | $q/p$ | $q/p^2$ | $(q = 1-p)$ |
| | | $f(y) = p\, q^{y-1}$ | $y \in \{1, ...\}$ | $1/p$ | $q/p^2$ | $(y = x+1)$ |
| **HyperGeo.** | HG$(n, A, B)$ | $f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$ | $x \in 0, \cdots, n$ | $n\,P$ | $n\,P\,(1-P)\frac{N-n}{N-1}$ | $(P = \frac{A}{A+B})$ |
| **Logistic** | Lo$(\mu, \beta)$ | $f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$ | $x \in \mathbb{R}$ | $\mu$ | $\pi^2\beta^2/3$ | |
| **Log Normal** | LN$(\mu, \sigma^2)$ | $f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$ | $x \in \mathbb{R}_+$ | $e^{\mu+\sigma^2/2}$ | $e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$ | |
| **Neg. Binom.** | NB$(\alpha, p)$ | $f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$ | $x \in \mathbb{Z}_+$ | $\alpha q/p$ | $\alpha q/p^2$ | $(q = 1-p)$ |
| | | $f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$ | $y \in \{\alpha, ...\}$ | $\alpha/p$ | $\alpha q/p^2$ | $(y = x+\alpha)$ |
| **Normal** | No$(\mu, \sigma^2)$ | $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ | $x \in \mathbb{R}$ | $\mu$ | $\sigma^2$ | |
| **Pareto** | Pa$(\alpha, \epsilon)$ | $f(x) = \alpha\, \epsilon^\alpha / x^{\alpha+1}$ | $x \in (\epsilon, \infty)$ | $\frac{\epsilon\,\alpha}{\alpha-1}$ | $\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ | |
| **Poisson** | Po$(\lambda)$ | $f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ | $x \in \mathbb{Z}_+$ | $\lambda$ | $\lambda$ | |
| **Snedecor** $F$ | $F(\nu_1, \nu_2)$ | $f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2}\, x\right]^{-\frac{\nu_1+\nu_2}{2}}$ | $x \in \mathbb{R}_+$ | $\frac{\nu_2}{\nu_2-2}$ | $\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ | |
| **Student** $t$ | $t(\nu)$ | $f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}$ | $x \in \mathbb{R}$ | $0$ | $\nu/(\nu-2)$ | |
| **Uniform** | Un$(a, b)$ | $f(x) = \frac{1}{b-a}$ | $x \in (a, b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | |
| **Weibull** | We$(\alpha, \beta)$ | $f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$ | $x \in \mathbb{R}_+$ | $\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$ | $\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$ | |