

Sta 711: Homework 3

Random variables

1. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1], \mathcal{B}, \lambda)$ for Lebesgue measure λ on the Borel sets of the unit interval. For $\omega \in \Omega$ define:

$$X_1(\omega) \equiv \min(\omega, 0.6) \quad X_2(\omega) \equiv \mathbf{1}_{(0,1/3]}(\omega) \quad X_3(\omega) \equiv \sqrt{\omega}$$

Plot each of the CDFs $F_k(x) \equiv \mathbb{P}[X_k \leq x]$, $x \in \mathbb{R}$, and describe explicitly the σ -algebras $\mathcal{F}_k := \sigma(X_k)$.

2. Let X be a random variable with CDF $F(x) := \mathbb{P}(X \leq x)$. Set $Y \equiv F(X)$. If X has a continuous distribution (*i.e.*, if F is a continuous function), show that Y is a random variable and that Y has a uniform distribution on $[0, 1]$. Warning: $F(x)$ may not be *strictly* increasing, and so may not be one-to-one; also it may not be differentiable.
3. A random variable Y is *real-valued* if $Y(\omega) \in \mathbb{R}$ for every $\omega \in \Omega$, and is *bounded* if there is a fixed finite number $0 \leq B < \infty$ for which $|Y(\omega)| \leq B$ for all $\omega \in \Omega$. Give an example of a real-valued random variable X that is *not* bounded.
4. Let X be a real valued random variable (so $\mathbb{P}[|X| < \infty] = 1$) with CDF $F(x)$. For each $\epsilon > 0$, construct a *bounded* random variable Y_ϵ such that

$$\mathbb{P}(X \neq Y_\epsilon) < \epsilon.$$

Measurable functions

5. Let $\Omega = \mathbb{R}$. Show that $\mathcal{S} := \{\emptyset, (-\infty, 0], (0, \infty), \Omega\}$ is a σ -algebra. Describe all functions $f : \Omega \rightarrow \mathbb{R}$ that are $\mathcal{S} \setminus \mathcal{B}$ -measurable.
6. If X is a real-valued random variable on any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then show that $|X|$ is also a random variable. Show by an example that the converse need not be true (Hint: A finite Ω will suffice)
7. Let $\Omega = \mathbb{R}$, and let $\mathcal{S}_0 \equiv \{\emptyset, \Omega\}$ be the trivial σ -algebra. Find all measurable functions $X : (\Omega, \mathcal{S}_0) \rightarrow (\mathbb{R}, \mathcal{B})$.
8. Let $\mathcal{F}_X := \sigma(X)$ be the σ -algebra generated by the function $X(\omega) = \omega^2$ on $\Omega = \mathbb{R}$. Is the set $A = (-\infty, 0]$ in \mathcal{F}_X ? How about $B = [-4, 4]$? Why?
9. Let $\{X_n, n \geq 0\}$ be real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfy

$$\limsup_{n \rightarrow \infty} X_n(\omega) = +\infty$$

for every $\omega \in \Omega$, and let $B < \infty$ be a real number. Prove that the integer-valued quantity

$$\tau(\omega) \equiv \inf\{n \geq 0 : X_n(\omega) \geq B\}$$

is a random variable.

Extra credit: Prove that X_τ is also a random variable.

Random Variables and σ -Algebras

10. All parts of this problem concern the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel sets, and $\mathbf{P} = \lambda$ Lebesgue measure. Let $\delta_n(\omega)$ be the n th bit in the binary expansion of ω , given by

$$\delta_n(\omega) := \lceil 1 + 2^n \omega \rceil \pmod{2}$$

where $\lceil x \rceil$ is the least integer $\geq x$, and set

$$\mathcal{F}_n := \sigma\{\delta_1, \dots, \delta_n\} = \sigma\{(0, j/2^n) : j = 0, \dots, 2^n\}.$$

- (a) Find a single real-valued random variable X on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathcal{F}_3 = \sigma(X)$.
- (b) True or False: If Y is any other random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathcal{F}_3 = \sigma(Y)$, then $Y = g(X)$ for some Borel measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$. Give a proof or counter-example.
- (c) Let Z be a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ for which $\mathcal{F} = \sigma(Z)$ (recall $\mathcal{F} = \mathcal{B}(\Omega)$, $\Omega = (0, 1]$, and $\mathbf{P} = \lambda$). True or false: For each $\omega_1 \neq \omega_2$, necessarily $Z(\omega_1) \neq Z(\omega_2)$. Explain.