Sta 711: Homework 9

Uniform Integrability

1. **True or false?** Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.

   (a) If $\{X_n,\ n \in \mathbb{N}\}$ is a uniformly integrable (UI) collection of random variables, then X_n is uniformly bounded in L_1.

   (b) Define a sequence $\{X_n\}$ of random variables on the unit interval with Lebesgue measure, $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, and $\mathsf{P} = \lambda$, by $X_n := \sqrt{n}\mathbf{1}_{(0,\frac{1}{n}]}$. Then $\{X_n\}$ is UI.

   (c) Let $\{X_n\}$ be a sequence of random variables for which $e^{|X_n|}$ is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}e^{|X_n|} \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

   (d) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}|X_n| \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

Characteristic Functions

2. Let X be a random variable, and define

$$\phi_X(\theta) := \mathsf{E}(e^{i\theta X}), \qquad \theta \in \mathbb{R}$$

   Show that $\phi_X(\theta)$ is uniformly continuous in $\mathbb{R}$.

3. Find the characteristic functions of the following random variables:

   (a) $W := c$[1] (The superscripts in (a)–(c) are footnote indicators, not exponents)

   (b) $X \sim \mathsf{Un}(a, b)$[2]

   (c) $Y \sim \mathsf{Ga}(\alpha, \lambda)$[3]

   (d) $Z_n = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha, \lambda)$

   What is the distribution of Z_n? What happens as $n \to \infty$?

4. The distribution of a random variable X is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist n iid random variables $\{Y_i\}$ such that X has the same distribution as $\sum_{i=1}^n Y_i$. Use characteristic functions to show that if $X \sim \mathsf{Po}(\lambda)$, then X is infinitely divisible.[4]

[1] A constant random variable with value $c \in \mathbb{R}$

[2] Uniform, on the interval $(a, b) \subset \mathbb{R}$

[3] Gamma, with rate parameterization— with pdf $f(y \mid \lambda) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y}/\Gamma(\alpha)$, $y > 0$.

[4] Hint: If $\{Y_i\}$ are independent with sum $Y_+ := \sum Y_i$, then $\phi_{Y_+}(\theta) = \prod \phi_{Y_i}(\theta)$ for all $\theta \in \mathbb{R}$.

5. Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathsf{P}(A_n) < 1$, $\forall n \in \mathbb{N}$. Show that $\mathsf{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathsf{P}(A_n \text{ i.o.}) = 1$ ("i.o." means "infinitely often", so the question concerns $\limsup A_n$). Give an example to show that the condition $\mathsf{P}(A_n) < 1$ cannot be dropped.

6. Let $\{A_n\}$ be a sequence of events with $\mathsf{P}(A_n) \to 1$ as $n \to \infty$. Prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$.

7. Let A_n be a sequence of events that all satisfy $\mathsf{P}(A_n) \geq \epsilon$ for some $\epsilon > 0$. Does there necessarily exist a subsequence $\{n_k \to \infty\}$ with $\mathsf{P}(\cap_k A_{n_k}) > 0$? Why or why not?

8. Let $\{X_n\}$ be non-negative iid random variables, with tail σ-field

$$\mathcal{T} := \bigcap_{n \in \mathbb{N}} \mathcal{F}'_n, \qquad \mathcal{F}'_n := \sigma\{X_m : \ m > n\}$$

Is the event

$$\begin{aligned} E &= \{\text{There exists } \epsilon > 0 \text{ such that } X_n > n\epsilon \text{ for infinitely-many } n\} \\ &= \bigcup_{\epsilon > 0} \bigcap_{n \geq 1} \bigcup_{m \geq n} \{\omega : \ X_m(\omega) > m\,\epsilon\} \end{aligned}$$

in $\mathcal{T}$? Prove or disprove it.

Express the probability $\mathsf{P}[E]$ in terms of the random variables' common distribution—for example, using their common CDF $F(x) := \mathsf{P}[X_n \leq x]$ or moments $\mathsf{E}[|X_n|^p]$ for some $p > 0$.