

Sta 711: Homework 10

Conditional Expectation

1. Let $\{N_t\}_{t \geq 0}$ be a homogeneous Poisson process with rate λ , so $N_0 = 0$ and for every $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_n < \infty$ the random variables $X_i := [N_{t_i} - N_{t_{i-1}}]$ for $1 \leq i \leq n$ are independent with marginal distributions $X_i \sim \text{Po}(\lambda(t_i - t_{i-1}))$. For $0 < s < t < \infty$ find the conditional expectations:

$$\mathbb{E}[N_s \mid N_t] =$$

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2. Let $\{X_1, X_2\} \stackrel{\text{iid}}{\sim} \text{Ex}(1)$ be iid unit-rate exponential random variables, $t > 0$. Find:
 - (a) $\mathbb{E}[X_1 \mid X_1 + X_2] =$
 - (b) $\mathbb{P}[X_1 < 3 \mid X_1 + X_2] =$
 - (c) $\mathbb{E}[X_1 \mid X_1 \wedge t] =$
 - (d) $\mathbb{E}[X_1 \mid X_1 \vee t] =$
3. Let $X, Y \in L_2(\Omega, \mathcal{F}, \mathbf{P})$ and suppose $\mathbb{E}[X \mid Y] = \phi(Y)$ for a monotonically decreasing Borel function $\phi: \mathbb{R} \rightarrow \mathbb{R}$. Prove that $\text{Cov}(X, Y) \leq 0$.
4. If $X \in L_2(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, show

$$\mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{H}])^2\right] \geq \mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{G}])^2\right]$$

In English, the MSE of the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ given the bigger σ -algebra \mathcal{G} is smaller than the MSE of the conditional expectation $\mathbb{E}[X \mid \mathcal{H}]$ given the smaller σ -algebra \mathcal{H} .

What does this imply for the trivial σ -algebra $\mathcal{H} = \{\emptyset, \Omega\}$?